Mathematics 1350H – Linear algebra I: Matrix algebra
TRENT UNIVERSITY, Summer 2015

SOLUTIONS TO THE FINAL EXAMINATION
Friday, 19 June, 2015

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Do parts **A** and **B**, and, if you wish, part **C**. Show all your work. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; ≤ 1 brain.

Part A. Do *all four* (4) of questions **1–4**. *[Subtotal = 64/100]*

1. Consider the following system of linear equations and its coefficient matrix $\mathbf{A}$:

$$\begin{array}{ccccccccccc} & & v & & & + & y & & & = & 2 \\ u & + & 4v & + & 5x & + & 4y & + & z & = & 15 \\ u & & & + & x & & & + & z & = & 3 \\ & & 2v & + & 2x & + & 2y & & & = & 6 \end{array} \qquad \text{and} \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 4 & 5 & 4 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 2 & 2 & 0 \end{bmatrix}$$

Note that $u = v = x = y = z = 1$ is a solution of this system of linear equations.

a. Without any calculations, determine how many solutions this system has. *[3]*

b. Use Gauss-Jordan reduction to find all the solutions of this system. *[10]*

c. Find the rank and nullity of $\mathbf{A}$. *[2]*

d. Find a basis for the column space of $\mathbf{A}$. *[2]*

e. Find a basis for the null space of $\mathbf{A}$. *[3]*

SOLUTIONS. **a.** Since the given system of linear equations has at least one solution, namely $u = v = x = y = z = 1$, it has either just one solution or many solutions. On the other hand, since it has five variables and only four equations, it must have either no solutions or many solutions. It follows that the system must have many solutions because that is the only common alternative. $\square$

b. We set up the corresponding augmented matrix and row-reduce away:

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 1 & 0 & 2 \\ 1 & 4 & 5 & 4 & 1 & 15 \\ 1 & 0 & 1 & 0 & 1 & 3 \\ 0 & 2 & 2 & 2 & 0 & 6 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 3 \\ 1 & 4 & 5 & 4 & 1 & 15 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 2 & 2 & 2 & 0 & 6 \end{array}\right]$$

$$\begin{array}{c} R_2 - R_1 \\ \Longrightarrow \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 3 \\ 0 & 4 & 4 & 4 & 0 & 12 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 2 & 2 & 2 & 0 & 6 \end{array}\right] \begin{array}{c} \frac{1}{4}R_2 \\ \Longrightarrow \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 2 & 2 & 2 & 0 & 6 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ \\ R_3 - R_2 \\ R_4 - 2R_2 \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \Longrightarrow \\ (-1)R_3 \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_3 \\ R_2 - R_3 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} \text{That is,} \\ u + z = 2, \\ v + y = 2, \\ \text{and } x = 1. \end{array}$$

If we set $y = s$ and $z = t$ for parameters s and t, then $u = 2 - z = 2 - t$, $v = 2 - y = 2 - s$, and $x = 1$. It follows that all the solutions of the given system of linear equations are given by

$$\begin{bmatrix} u \\ v \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2-t \\ 2-s \\ 1 \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where $s, t \in \mathbb{R}$. □

c. Since the reduced matrix in the solution to **b** above has three non-zero rows, $\text{rank}(\mathbf{A}) = 3$. Since $\mathbf{A}$ has five columns, it follows that $\text{nullity}(\mathbf{A}) = 5 - 3 = 2$. □

d. The leading entries of the non-zero rows of the reduced matrix in the solution to **b** are in the first three columns. The corresponding columns of the original matrix are a basis for its column space, so

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \\ 2 \end{bmatrix} \right\}$$

is a basis for the column space of $\mathbf{A}$. □

e. The null space of $\mathbf{A}$ is the set of solutions to the homogeneous system $\mathbf{Ax} = \mathbf{0}$. We can find these by setting up the corresponding augmented matrix and reducing it just as in the solution to **b**, noting that the right-hand side will remain $\mathbf{0}$ throughout:

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 4 & 5 & 4 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_3 \\ \\ \Longrightarrow \\ \end{array} \quad \cdots \quad \begin{array}{c} R_1 - R_3 \\ R_2 - R_3 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The reduced matrix boils to the equations $u + z = 0$, $v + y = 0$, and $x = 0$; setting $y = s$ and $z = t$ for parameters s and t, we get $u = -z = -t$ and $v = -y = -t$. Ths the homogeneous system has solutions:

$$\begin{bmatrix} u \\ v \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ -s \\ 1 \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where $s,\ t \in \mathbb{R}$. It follows that $\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for null($\mathbf{A}$). [Alternatively, you could just note that these were the vectors corresponding to the paramaters in the solution to **b** and explain why they serve as the basis for the null space.] ■

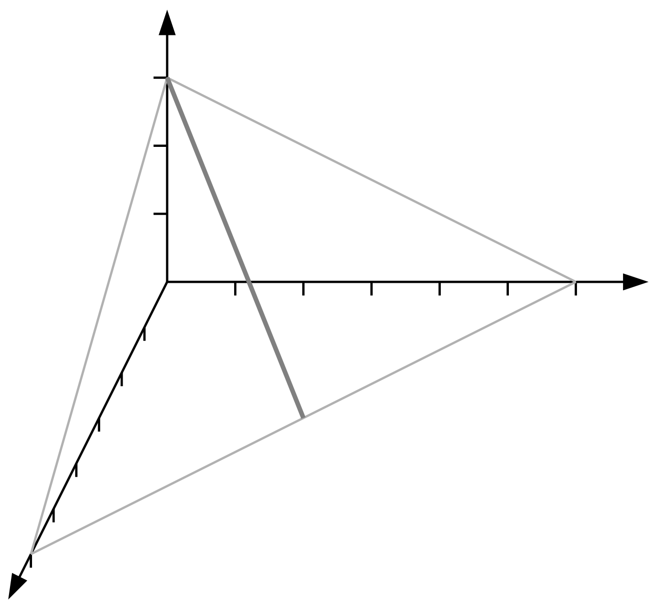
2. Consider the plane and line in $\mathbb{R}^3$ given by the linear equation $x + y + 2z = 6$ and the parametric equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $t \in \mathbb{R}$, respectively.

a. Sketch the line. *[3]*

b. Sketch the plane. *[2[*

c. Determine whether or not the line is contained in the plane. *[5]*

SOLUTIONS. **a & b.** Here is a sketch of the part of the plane and line in the first octant. Note that the intercepts of the plane are the points $(6, 0, 0)$, $(0, 6, 0)$, and $(0, 0, 3)$; note also that the line meets the plane $z = 0$ at the point $(3, 3, 0)$ when $t = 1$, and the z-axis at the point $(0, 0, 3)$ when $t = -2$.



□

c. *[The geometric way.]* Note that the points $(3, 3, 0)$ and $(0, 0, 3)$ on the line noted above both satisfy the equation for the plane, and hence are on the plane, too. Since two different points on the line are also on the plane, the entire line must be contained in the plane. □

c. *[The algebraic way.* Every point on the line has the form $(x, y, z) = (2 + t, 2 + t, 1 - t)$ for some $t \in \mathbb{R}$. Plugging these coordinates into the equation of the plane yields

$$x + y + 2z = (2 + t) + (2 + t) + 2(1 - t) = (2 + 2 + 2) + (t + t - 2t) = 6 + 0 = 6,$$

so every point on the line satisfies the equation of the plane. It follows that every point on the line is also on the plane, *i.e.* the line is contained in the plane. $\square$

c. *[The vector way.]* The base vector of the line takes us from the origin to the point $(2,2,1)$, which satsifies the equation of the plane. The direction vector of the line, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and the normal vector of the plane, $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, are orthogonal to each other because

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 1 \cdot 1 + 1 \cdot 1 + (-1) \cdot 2 = 1 + 1 - 2 = 0,$$

so the line is parallel to the plane. A line parallel to a plane that has a point in common with the plane must be contained in the plane. $\blacksquare$

3. Let $\mathbf{A} = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 8 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

a. Compute $\mathbf{A}^{-1}$, if it exists. *[10]*
b. What are the rank and nullity of $\mathbf{A}$? Why? *[2]*
c. How many solutions $\mathbf{x}$ are there to $\mathbf{Ax} = \mathbf{0}$? *[2]*
d. Compute $|\mathbf{A}|$. *[5]*

SOLUTIONS. **a.** We set up the "super-augmented" matrix $[\mathbf{A}|\mathbf{I}]$ and let slip the dogs of reduction:

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 2 & 1 & 0 & 0 \\ 3 & 8 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \leftrightarrow R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 3 & 8 & 2 & 0 & 1 & 0 \\ 2 & 3 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 5 & -1 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 & 0 & -2 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 \leftrightarrow R_3 \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 5 & -1 & 0 & 1 & -3 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ \Longrightarrow \\ R_3 - 5R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & -5 & 1 & 7 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ (-1)R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 & -1 & -7 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_3 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 1 & 10 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 & -1 & -7 \end{array}\right], \text{ so } \mathbf{A}^{-1} = \begin{bmatrix} -6 & 1 & 10 \\ 1 & 0 & -2 \\ 5 & -1 & -7 \end{bmatrix}. \quad \square$$

b. Since applying the Gauss-Jordan algorithm to $\mathbf{A}$, as in the solution to part **a** above, leaves three non-zero rows, $\text{rank}(\mathbf{A}) = 3$. As, by the solution to part **a**, $\mathbf{A}$ is invertible, we must have $\text{nullity}(\mathbf{A}) = 0$. $\square$

c. Since $\mathbf{A}$ is invertible, $\mathbf{Ax} = \mathbf{b}$ has an unique solution for every $\mathbf{b} \in \mathbb{R}^3$. $\square$

d. The only row operations that changed the determinant in the reduction of $\mathbf{A}$ to $\mathbf{I}_2$ in the solution to **a** above were the two row swaps, $R_1 \leftrightarrow R_3$ at the first step and $R_2 \leftrightarrow R_3$ at the third step, and the multiplication of row 3 by -1 at the fifth step. Each of these multiplied the determinant by -1, so $(-1)^3|\mathbf{A}| = |\mathbf{I}_2| = 1$. It follows that $|\mathbf{A}| = \frac{1}{(-1)^3} = \frac{1}{-1} = -1$. $\blacksquare$

4. Consider the subspace $W = \text{Span}\left\{ \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \right\}$ of $\mathbb{R}^4$.

- **a.** Find a basis for W. *[10]*
- **b.** Find an orthogonal basis for W. *[5]*

SOLUTIONS. **a.** We put the vectors of the given spanning set in as the columns of a matrix and row-reduce the matrix:

$$\begin{bmatrix}0&0&0&1&1&1\\0&1&1&0&0&1\\1&0&1&0&1&0\\1&1&0&1&0&0\end{bmatrix} \overset{R_1\leftrightarrow R_3}{\Longrightarrow} \begin{bmatrix}1&0&1&0&1&0\\0&1&1&0&0&1\\0&0&0&1&1&1\\1&1&0&1&0&0\end{bmatrix}$$

$$\underset{R_4-R_1}{\Longrightarrow} \begin{bmatrix}1&0&1&0&1&0\\0&1&1&0&0&1\\0&0&0&1&1&1\\0&1&-1&1&-1&0\end{bmatrix} \underset{R_4-R_2}{\Longrightarrow} \begin{bmatrix}1&0&1&0&1&0\\0&1&1&0&0&1\\0&0&0&1&1&1\\0&0&-2&1&-1&-1\end{bmatrix}$$

$$\underset{R_3\leftrightarrow R_4}{\Longrightarrow} \begin{bmatrix}1&0&1&0&1&0\\0&1&1&0&0&1\\0&0&-2&1&-1&-1\\0&0&0&1&1&1\end{bmatrix} \underset{-\frac{1}{2}R_3}{\Longrightarrow} \begin{bmatrix}1&0&1&0&1&0\\0&1&1&0&0&1\\0&0&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\0&0&0&1&1&1\end{bmatrix}$$

$$\underset{\substack{R_1-R_3\\R_2-R_3}}{\Longrightarrow} \begin{bmatrix}1&0&0&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\0&1&0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\0&0&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\0&0&0&1&1&1\end{bmatrix} \underset{\substack{R_1-\frac{1}{2}R_4\\R_2-\frac{1}{2}R_4\\R_3+\frac{1}{2}R_4}}{\Longrightarrow} \begin{bmatrix}1&0&0&0&0&-1\\0&1&0&0&-1&0\\0&0&1&0&1&1\\0&0&0&1&1&1\end{bmatrix}$$

Since the leading 1s of non-zero rows are in columns one to four of the reduced matrix, the first four vectors of the original spanning set form a basis for W. That is,

$$\left\{ \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix} \right\}$$

is a basis for W. □

b. *(The way of pain.)* To obtain an orthogonal basis for W, we apply the Gram-Schmidt orthogonalization process to the basis for W obtained in the solution to **a**:

$$\mathbf{b}_1 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

$$\mathbf{b}_2 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix} - \frac{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\\1\end{bmatrix}}{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\\1\end{bmatrix}} \begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$

$$\mathbf{b}_3 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} - \frac{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\\1\end{bmatrix}}\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{\begin{bmatrix}0\\1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}0\\1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}}\begin{bmatrix}0\\1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}0\\1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix} = \begin{bmatrix}0\\\frac{2}{3}\\\frac{2}{3}\\-\frac{2}{3}\end{bmatrix}$$

$$\mathbf{b}_4 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix} - \frac{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\\1\end{bmatrix}}{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\\1\end{bmatrix}}\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{\begin{bmatrix}0\\1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\\1\end{bmatrix}}{\begin{bmatrix}0\\1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}0\\1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}}\begin{bmatrix}0\\1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix} - \frac{\begin{bmatrix}0\\\frac{2}{3}\\\frac{2}{3}\\-\frac{2}{3}\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\\1\end{bmatrix}}{\begin{bmatrix}0\\\frac{2}{3}\\\frac{2}{3}\\-\frac{2}{3}\end{bmatrix}\cdot\begin{bmatrix}0\\\frac{2}{3}\\\frac{2}{3}\\-\frac{2}{3}\end{bmatrix}}\begin{bmatrix}0\\\frac{2}{3}\\\frac{2}{3}\\-\frac{2}{3}\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}0\\1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix} - \left(-\frac{1}{2}\right)\begin{bmatrix}0\\\frac{2}{3}\\\frac{2}{3}\\-\frac{2}{3}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

Thus $\left\{ \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}, \begin{bmatrix}0\\\frac{2}{3}\\\frac{2}{3}\\-\frac{2}{3}\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \right\}$ is an orthogonal basis for W. $\square$

b. *(The cheap way.)* Since the basis obtained in the solution to part **a** has four elements, W has dimension 4; since it is a 4-dimensional subspace of the 4-dimensional vector space $\mathbb{R}^4$, it follows that $W = \mathbb{R}^4$. The standard basis of $\mathbb{R}^4$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\} = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$, is known to be orthogonal ... $\blacksquare$

Part B. Do *any three* (3) of questions **5–10**. *[Subtotal = 36/100]*

5. Determine whether each of the following is a subspace of $\mathbb{R}^2$ or not. *[12 = 3 × 4 each]*

a. $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, xy = 0 \right\}$ **b.** $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, \cos(x+y) = 1 \right\}$

c. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, (x+y+\pi)^2 = (x-y+\pi)^2 \right\}$

SOLUTIONS. **a.** U is not a subspace of $\mathbb{R}^2$ because it is not closed under vector addition: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are both in U because $1 \cdot 0 = 0 = 0 \cdot 1$, but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in U because $1 \cdot 1 = 1 \neq 0$. □

b. V is not a subspace of $\mathbb{R}^2$ because it is not closed under multiplication by scalars: $\begin{bmatrix} \pi \\ \pi \end{bmatrix}$ is in V because $\cos(\pi+\pi) = \cos(2\pi) = 1$, but $\frac{1}{2}\begin{bmatrix} \pi \\ \pi \end{bmatrix} = \begin{bmatrix} \pi/2 \\ \pi/2 \end{bmatrix}$ is not in V because $\cos\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \cos(\pi) = -1 \neq 1$. □

c. W is not a subspace of $\mathbb{R}^2$ because it is not closed under vector addition: $\begin{bmatrix} -\pi \\ 1 \end{bmatrix}$ is in W because $(-\pi+1+\pi)^2 = 1^2 = 1 = (-1)^2 = (-\pi-1-\pi)^2$ and $\begin{bmatrix} \pi \\ 0 \end{bmatrix}$ is in W because $(\pi+0+\pi)^2 = 4\pi^2 = (\pi-0+\pi)^2$, but $\begin{bmatrix} -\pi \\ 1 \end{bmatrix} + \begin{bmatrix} \pi \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not in W because $(0+1+\pi)^2 = 1+2\pi+\pi^2 \neq 1-2\pi+\pi^2 = (0-1+\pi)^2$ as $2\pi \neq -2\pi$. ■

6. Recall that $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{O}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Find a 2×2 matrix $\mathbf{X}$ different from $\mathbf{I}_2$ which satisfies the equation $\mathbf{X}^2 - 2\mathbf{X} + \mathbf{I}_2 = \mathbf{O}_2$, or show that $\mathbf{I}_2$ is the only matrix that satisfies the equation. *[12]*

SOLUTION. $\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is a matrix other than $\mathbf{I}_2$ that satisfies the equation:

$$\begin{aligned} \mathbf{X}^2 - 2\mathbf{X} + \mathbf{I}_2 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 - 2\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-2+1 & 2-2+0 \\ 0-0+0 & 1-2+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}_2 \qquad ■ \end{aligned}$$

7. Find the shortest distance from the point $(3,3,1)$ to the line given by the parametric equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. *[12]*

Hint: There can be no objection to removing a projection to get orthogonality!

SOLUTION. The shortest distance from a point to a line is the length of the vector from the point to the line that is perpendicular to the line. We first find a vector from the given point $(3, 3, 1)$ to the base point of the line, $(0, 2, 2)$:

$$\begin{bmatrix} 3-0 \\ 3-2 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

The projection of this vector onto the direction vector of the line,

$$\text{proj}_{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}\left(\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}\right) = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{2}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

is the component that is parallel to the line. Removing it leaves the vector perpendicular to the line from the given point to the line:

$$\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

The length of this vector,

$$\left\|\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}\right\| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3,$$

is the distance from the given point to the given line. ■

8. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that $T\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$, $T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, and $T\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$. Find the matrix $[T]$ of T (that is, such that $[T]\mathbf{x} = T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$). *[12]*

SOLUTION. We will use the fact that $[T] = [T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)]$. Observe that:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$$

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right)$$

Since T is a linear transformation, it follows that:

$$T(\mathbf{e}_1) = \frac{1}{2}T\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}2\\0\\2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\2\\0\end{bmatrix} = \begin{bmatrix}2\\1\\1\end{bmatrix}$$
$$T(\mathbf{e}_2) = \frac{1}{2}T\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}2\\2\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\2\\2\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix}$$
$$T(\mathbf{e}_3) = \frac{1}{2}T\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}2\\0\\2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\2\\2\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix}$$

Thus $[T] = [T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)] = \begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$. ■

9. Give examples in each case, or explain why none exist, of 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ which have inverses such that:

a. $\mathbf{A} - \mathbf{B}$ has no inverse. *[3]* **b.** $\mathbf{A} - \mathbf{B}$ has an inverse. *[3]*
c. $\mathbf{AB}$ has no inverse. *[3]* **d.** $\mathbf{AB}$ has an inverse. *[3]*

Solutions. **a.** Let $\mathbf{A} = \mathbf{B} = \mathbf{I}_3$. Then $\mathbf{A}$ and $\mathbf{B}$ are invertible, with $\mathbf{A}^{-1} = \mathbf{B}^{-1} = \mathbf{I}_3$, but $\mathbf{A} - \mathbf{B} = \mathbf{I}_3 - \mathbf{I}_3 = \mathbf{O}_3$ is not invertible. □

b. Let $\mathbf{A} = 2\mathbf{I}_3$ and $\mathbf{B} = \mathbf{I}_3$. Then $\mathbf{A}$ and $\mathbf{B}$ are invertible, with $\mathbf{A}^{-1} = \frac{1}{2}\mathbf{I}_3$ and $\mathbf{B}^{-1} = \mathbf{I}_3$, and $\mathbf{A} - \mathbf{B} = 2\mathbf{I}_3 - \mathbf{I}_3 = \mathbf{I}_3$ is also invertible. □

c. There is no such example. If $\mathbf{A}$ and $\mathbf{B}$ have inverses, then so does $\mathbf{AB}$: recall from class that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. □

d. Let $\mathbf{A} = \mathbf{B} = \mathbf{I}_3$. Then $\mathbf{A}$ and $\mathbf{B}$ are invertible, with $\mathbf{A}^{-1} = \mathbf{B}^{-1} = \mathbf{I}_3$, and $\mathbf{AB} = \mathbf{I}_3\mathbf{I}_3 = \mathbf{I}_3$ also has inverse $\mathbf{I}_3$. ■

10. Let $\mathbf{B} = \begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}$.

a. Find all the eigenvalues of $\mathbf{B}$. *[5]*
b. Find a nonzero eigenvector for each eigenvalue of $\mathbf{B}$. *[5]*
c. Without computing it, what is $|\mathbf{B}|$? *[2]*

Solutions. **a.** We try to solve the equation $|\mathbf{B} - \lambda\mathbf{I}_3| = 0$ for λ. Expanding along the first row:

$$\begin{aligned}
|\mathbf{B} - \lambda\mathbf{I}_3| &= \left|\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix} - \lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right| = \begin{vmatrix}-\lambda&1&0\\1&-\lambda&1\\0&1&-\lambda\end{vmatrix} \\
&= +(-\lambda)\begin{vmatrix}-\lambda&1\\1&-\lambda\end{vmatrix} - 1\begin{vmatrix}1&1\\0&-\lambda\end{vmatrix} + 0\begin{vmatrix}1&-\lambda\\0&1\end{vmatrix} \\
&= -\lambda\left[(-\lambda)(-\lambda) - 1\cdot 1\right] - \left[1(-\lambda) - 0\cdot 1\right] + 0 \\
&= -\lambda^3 + \lambda + \lambda = 2\lambda - \lambda^3 = \lambda\left(2 - \lambda^2\right) = \lambda\left(\sqrt{2} - \lambda\right)\left(\sqrt{2} + \lambda\right)
\end{aligned}$$

Thus $|\mathbf{B} - \lambda \mathbf{I}_3| = 0$ if $\lambda = 0$, $\sqrt{2}$, or $-\sqrt{2}$, so these are the eigenvalues of $\mathbf{B}$. □

b. We need to find a solution $\mathbf{x} \neq \mathbf{0}$ to $(\mathbf{B} - \lambda \mathbf{I}_3)\,\mathbf{x} = \mathbf{0}$ for each of $\lambda = 0$, $\sqrt{2}$, and $-\sqrt{2}$. We set up and reduce the augmented matrix $[\mathbf{B} - \lambda \mathbf{I}|\mathbf{0}]$ for the equation in each case:

$$\lambda = 0: \quad \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \underset{}{\overset{R_1 \leftrightarrow R_2}{\Longrightarrow}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \underset{R_3 - R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This corresponds to the system of equations $x + z = 1$ and $y = 0$. Setting $x = 1 \neq 0$, we must have $z = -1$, which gives us the eigenvector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ for the eigenvalue $\lambda = 0$.

$$\lambda = \sqrt{2}: \quad \left[\begin{array}{ccc|c} -\sqrt{2} & 1 & 0 & 0 \\ 1 & -\sqrt{2} & 1 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & -\sqrt{2} & 1 & 0 \\ -\sqrt{2} & 1 & 0 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \end{array}\right]$$

$$\underset{R_2 + \sqrt{2}R_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & -\sqrt{2} & 1 & 0 \\ 0 & -1 & \sqrt{2} & 0 \\ 0 & 1 & -\sqrt{2} & 0 \end{array}\right] \underset{(-1)R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & -\sqrt{2} & 1 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & 1 & -\sqrt{2} & 0 \end{array}\right]$$

$$\underset{R_3 - R_2}{\overset{R_1 + \sqrt{2}R_2}{\Longrightarrow}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This corresponds to the system of equations $x - z = 0$ and $y - \sqrt{2}z = 0$.

Setting $x = 1 \neq 0$ gives $z = 1$, and hence $y = \sqrt{2}$, which gives us the eigenvector $\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$ for the eigenvalue $\lambda = \sqrt{2}$.

$$\lambda = -\sqrt{2}: \quad \left[\begin{array}{ccc|c} \sqrt{2} & 1 & 0 & 0 \\ 1 & \sqrt{2} & 1 & 0 \\ 0 & 1 & \sqrt{2} & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & \sqrt{2} & 1 & 0 \\ \sqrt{2} & 1 & 0 & 0 \\ 0 & 1 & \sqrt{2} & 0 \end{array}\right]$$

$$\underset{R_2 - \sqrt{2}R_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & \sqrt{2} & 1 & 0 \\ 0 & -1 & -\sqrt{2} & 0 \\ 0 & 1 & \sqrt{2} & 0 \end{array}\right] \underset{(-1)R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & \sqrt{2} & 1 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & 1 & \sqrt{2} & 0 \end{array}\right]$$

$$\underset{R_3 - R_2}{\overset{R_1 - \sqrt{2}R_2}{\Longrightarrow}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This corresponds to the system of equations $x - z = 0$ and $y + \sqrt{2}z = 0$.

Setting $x = 1 \neq 0$ gives $z = 1$, and hence $y = -\sqrt{2}$, which gives us the eigenvector $\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$ for the eigenvalue $\lambda = -\sqrt{2}$. □

c. Since, by the solution to part **a**, 0 is an eigenvalue of $\mathbf{B}$, $|\mathbf{B}| = |\mathbf{B} - 0\mathbf{I}_3| = 0$. ■

[Total = 100]

Part C. *Bonus!*

0. Write an original little poem about linear algebra or mathematics in general. *[1]*

SOLUTION. It's supposed to be original, so it all up to you! ■

~~FRANKENFURTER, IT'S ALL OVER~~!
GO FORTH AND ENJOY THE SUMMER!