Trent University

# MATH 1350H Test

3 June, 2014

Time: 50 minutes

Name: Solutions

Student Number: 0.577215

| Question | Mark | |
|---|---|---|
| 1 | ____ | |
| 2 | ____ | |
| 3 | ____ | |
| 4 | ____ | |
| **Bonus** | ____ | |
| **Total** | ____ | /40 |

**Instructions**

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

**1.** Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

**a.** Find the angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$.

**b.** Determine whether the lines given by $\mathbf{x} = \mathbf{u} + s\mathbf{v}$ and $\mathbf{x} = \mathbf{v} + t\mathbf{w}$ intersect or not.

**c.** Find a non-zero vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

SOLUTIONS. **a.** We'll use the formula $\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$.

$$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\left\|\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\| \left\|\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\|} = \frac{1\cdot 1 + 0\cdot 1 + 1\cdot 0}{\sqrt{1^2+0^1+1^2}\,\sqrt{1^2+1^2+0^2}} = \frac{1}{\sqrt{2}\,\sqrt{2}} = \frac{1}{2}$$

Since $\cos(\theta) = \frac{1}{2}$, $\theta = 60° = \frac{\pi}{3}$ radians. □

**b.** For the lines to intersect, they must have a common point, which means that there must be scalars $s$ and $t$ such that $\mathbf{u} + s\mathbf{v} = \mathbf{x} = \mathbf{v} + t\mathbf{w}$, *i.e.* such that

$$\begin{bmatrix} 1+s \\ s \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ t \end{bmatrix}.$$

Looking at the second and third coordinates, one can see that this requires $s = t = 1$, but then we get $2 = 1 + s = 1$ in the first coordinate, which is impossible. It follows that the two lines do not intersect. □

**c.** We need to find a vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x}\cdot\mathbf{u} = 0$ and $\mathbf{x}\cdot\mathbf{v} = 0$, *i.e.* such that

$$\mathbf{x}\cdot\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = x + z = 0 \quad \text{and} \quad \mathbf{x}\cdot\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = x + y = 0\,.$$

If we set $x = 1$ to ensure that $\mathbf{x} \neq \mathbf{0}$, it is easy to see that we need to have $y = z = -1$. Thus $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is a vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. ■

NOTE: One could also find such a vector by taking the cross-product of $\mathbf{u}$ and $\mathbf{v}$, *e.g.* $\mathbf{n} = \mathbf{u}\times\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j} + \mathbf{k} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. This will, however it's done, give a multiple of the answer obtained above.

**2.** Consider the following system of linear equations:
$$\begin{array}{rcrcrcr} x & & & + & z & = & 4 \\ x & + & 2y & & & = & 5 \\ 2x & + & 2y & + & kz & = & 12 \end{array}$$

**a.** Find all the solutions, if any, of this system for *one* (1) of $\left\{ \begin{array}{ll} i. & k=1 \\ ii. & k=2 \end{array} \right.$. *[8]*

**b.** Use your answer to **a** to determine whether $\begin{bmatrix} 4 \\ 5 \\ 12 \end{bmatrix} \in \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ k \end{bmatrix} \right\}$. *[2]*

Solutions. **a.** To avoid doing the same things twice in this solution, we set up the augmented matrix with a generic $k$ and apply the Gauss-Jordam method:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 1 & 2 & 0 & 5 \\ 2 & 2 & k & 12 \end{array}\right]$$

$$\begin{array}{r} \Longrightarrow \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 2 & -1 & 1 \\ 0 & 2 & k-2 & 4 \end{array}\right]$$

$$\begin{array}{r} \Longrightarrow \\ \frac{1}{2}R_2 \\ \phantom{x} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & k-2 & 4 \end{array}\right]$$

$$\begin{array}{r} \Longrightarrow \\ R_3 - 2R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & k-1 & 3 \end{array}\right]$$

In case $i$, with $k = 1$, we'd be done at this point, since the third row would be $[0 \quad 0 \quad 0 \quad | \quad 3]$, indicating there is no solution; otherwise, in case $ii$, $k - 1 = 2 - 1 = 1$, and we continue:

$$\begin{array}{r} R_1 - R_3 \\ R_2 + \frac{1}{2}R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

It follows that in case $i$, where $k = 1$, there is no solution to the given system of linear equations, while in case $ii$, where $k = 2$, there is exactly one solution, namely $x = 1$, $y = 2$, and $z = 3$. □

**b.** By definition, $\begin{bmatrix} 4 \\ 5 \\ 12 \end{bmatrix} \in \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ k \end{bmatrix} \right\}$ exactly when there are scalars – let's call them $x$, $y$, and $z$ – such that $x \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ k \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 12 \end{bmatrix}$, that is, if the given system of equations has a solution. In case $i$, where $k = 1$, there is no solution and so $\begin{bmatrix} 4 \\ 5 \\ 12 \end{bmatrix}$ is not in the span; in case $ii$, where $k = 2$, there is a solution, and so $\begin{bmatrix} 4 \\ 5 \\ 12 \end{bmatrix}$ is in the span. ■

**3.** Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

**a.** Show that if the $n \times n$ matrix $\mathbf{A}$ has an inverse and $c \neq 0$, then $c\mathbf{A}$ has an inverse.

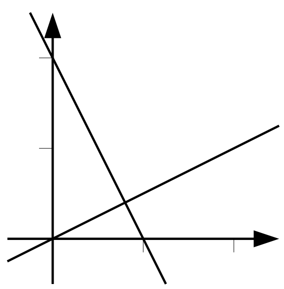
**b.** Find a vector-parametric equation for the plane $x - y + z = 3$.

**c.** Sketch the lines $x + 2y = 2$ and $4x - 2y = 0$ and determine the angle between them.

SOLUTIONS. **a.** If $\mathbf{A}^{-1}$ exists and $c \neq 0$, then $(c\mathbf{A})\left(\frac{1}{c}\mathbf{A}^{-1}\right) = c\frac{1}{c}\mathbf{A}\mathbf{A}^{-1} = 1\mathbf{I}_n = \mathbf{I}_n$, so $c\mathbf{A}$ has the inverse $(c\mathbf{A})^{-1} = \frac{1}{c}\mathbf{A}^{-1}$. □

**b.** The $x$-, $y$-, and $z$-intercepts of the plane $x - y + z = 3$ are the points $(3, 0, 0)$, $(0, -3, 0)$, and $(0, 0, 3)$, respectively. These points form a triangle since they are on different axes. (Note that each point is outside of the plane defined by the axes it is not the intercept for, and the other two points are in that plane.) We can use the position vector of one of the points as the base vector for the vector-parametric equation of the plane defined by the three points (*i.e.* by $x - y + z = 3$), say $\mathbf{p} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, and use the vectors from that point to the other two as the direction vectors parallel to the plane: $\mathbf{u} = \begin{bmatrix} 0-3 \\ -3-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0-3 \\ 0-0 \\ 3-0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$. Thus $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$ is a vector-parametric equation for the plane $x - y + z = 3$. □

**c.** Here is a crude sketch:



As the sketch suggests, the angle between the lines is $90° = \frac{\pi}{2}$ radians. This can be verified by solving for $y$ in each equation, which gives $y = -\frac{1}{2}x + 1$ and $y = 2x$, respectively, and noting that the slopes of the two lines are each others' negative reciprocals. Alternatively, one can read off normal vectors for each line from the original equations, giving $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$, and observe that $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 1 \cdot 4 + 2 \cdot (-2) = 4 - 4 = 0$, which means that the normal vectors are orthogonal to each other, and hence that the lines are perpendicular. ■

**4.** Find the inverse matrix of $\begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 9 \\ 0 & 1 & 2 \end{bmatrix}$. *[10]*

Solutions. We set up the appropriate super-augmented matrix and apply the Gauss-Jordan method:

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 1 & 3 & 9 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2 \\ \Longrightarrow \\ \phantom{x} \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 9 & 0 & 1 & 0 \\ 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ \phantom{x} \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 9 & 0 & 1 & 0 \\ 0 & -2 & -12 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 \leftrightarrow R_3 \\ \phantom{x} \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 9 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & -2 & -12 & 1 & -2 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 - 3R_2 \\ \Longrightarrow \\ R_3 + 2R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & -3 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -8 & 1 & -2 & 2 \end{array}\right] \begin{array}{c} \phantom{x} \\ \Longrightarrow \\ -\frac{1}{8}R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & -3 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

$$\begin{array}{c} R_1 - 3R_3 \\ R_2 - 2R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{8} & \frac{1}{4} & -\frac{9}{4} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

The inverse matrix therefore exists and

$$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 9 \\ 0 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{1}{4} & -\frac{9}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

You can check that the inverse is correct by multiplying the original matrix by it. ■

*[Total=40]*

**Bonus.** A chip truck sells fries, cans of pop, and sandwiches. Any order of fries costs as much as any other, and similarly for cans of pop and sandwiches, respectively. $A$ buys two orders of fries, two cans of pop, and a sandwich, which costs \$10.00; $B$ buys two orders of fries, a can of pop, and a sandwich, which costs \$8.50; $C$ buys an order of fries, two cans of pop, and a sandwich, which costs \$8.00; and $D$ buys two orders of fries, two cans of pop, and two sandwiches. What does $D$'s purchase cost? *[1]*

Solution. The difference between $A$'s purchase and $B$'s purchase tells us that a can of pop costs \$1.50, and the difference between $A$'s purchase and $C$'s purchase tells us that an order of fries costs \$2.00. Plugging this information into the purchases of $A$, $B$, or $C$ lets us conclude that a sandwich costs \$3.00. Thus $D$'s purchase costs $2 \cdot 2 + 2 \cdot 1.5 + 2 \cdot 3 =$ \$13.00. ■