Mathematics 1350H – Linear algebra I: Matrix algebra
TRENT UNIVERSITY, Summer 2014

Solutions to Assignment #5
Determinants by way of Gauss-Jordan reduction

Given a square matrix $\mathbf{A}$, we can compute a number called the *determinant* of $\mathbf{A}$, usually denoted by $|\mathbf{A}|$ or $\det(\mathbf{A})$, that gives a lot of information about $\mathbf{A}$. For example, $|\mathbf{A}| \neq 0$ exactly when $\mathbf{A}^{-1}$ exists. One problem with the usual definition of determinants [see §4.2 in the text], which works by reducing the determinant of an $n \times n$ matrix to an alternating sum of determinants of n different $(n-1) \times (n-1)$ sub-matrices, is that computing them this way is a *lot* of work unless $\mathbf{A}$ is a pretty small matrix or has a lot of 0s. (Heck, it's a pain even for 3×3 matrices with the usual definition, as we saw in computing cross-products of vectors in $\mathbb{R}^3$.) In this assignment, we will be looking at a method to compute the determinant of a matrix using the Gauss-Jordan method.

The determinant of an $n \times n$ matrix $\mathbf{A}$ satisfies the following rules:

i. The identity matrix has determinant equal to 1, *i.e.* $|\mathbf{I}_n| = 1$.

ii. If you exchange the ith and jth row of $\mathbf{A}$ to get the matrix $\mathbf{B}$, then $|\mathbf{B}| = -|\mathbf{A}|$.

iii. If you multiply the ith row of $\mathbf{A}$ by a constant c to get the matrix $\mathbf{C}$, then $|\mathbf{C}| = c|\mathbf{A}|$.

iv. If you add a multiple of any row of $\mathbf{A}$ to a different row of $\mathbf{A}$ to get the matrix $\mathbf{D}$, then $|\mathbf{D}| = |\mathbf{A}|$.

v. Taking the transpose of $\mathbf{A}$ doesn't change the determinant. That is, $|\mathbf{A}^T| = |\mathbf{A}|$.

If you really wanted to, by the way, you could actually use this collection of rules as the definition of the determinant of a matrix. It's pretty cumbersome as a definition, but it does provide a much more efficient way to compute the determinant of even a modestly large matrix. It also makes it easier to see why $\mathbf{A}$ is invertible if and only if $|\mathbf{A}| \neq 0$: both are equivalent to the matrix being reducible to $\mathbf{I}_n$ using the Gauss-Jordan method.

1. In both **a** and **b** use the Gauss-Jordan method to put the matrix $\mathbf{A}$ in reduced row-echelon form, and then apply rules *i* – *v* to work out $|\mathbf{A}|$.

a. $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ *[2]*

b. $\mathbf{A} = \begin{bmatrix} 0 & 3 & 6 \\ 2 & 4 & 5 \\ 4 & 7 & 0 \end{bmatrix}$ *[3]*

SOLUTIONS. **a.** First, we apply the Gauss-Jordan method:

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \underset{\Longrightarrow}{\tfrac{1}{2}R_1} \begin{bmatrix} 1 & \frac{3}{2} \\ 4 & 5 \end{bmatrix} \underset{R_2 - 4R_1}{\Longrightarrow} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & -1 \end{bmatrix} \underset{(-1)R_2}{\Longrightarrow} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} \underset{\Longrightarrow}{R_1 - \tfrac{3}{2}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Second, we check how the row operations involved changed the determinant of $\mathbf{A}$. Note that since we never exchanged rows, rule *ii* does not apply. Otherwise, we subtracted a multiple of one row from another twice, which by rule *iv* does not change the determinant, and multiplied a row by a constant twice, which by rule *iii* multiplies the determinant by that constant. Thus $\left(\frac{1}{2}\right)(-1)|\mathbf{A}| = |\mathbf{I}_2| = 1$, the last equality being rule i. It follows that $|\mathbf{A}| = \frac{1}{\left(\frac{1}{2}\right)(-1)} = -2$.

Note that this answer agrees with that given by the formula for the determinant of a 2×2 matrix, $|\mathbf{A}| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2 \cdot 5 - 3 \cdot 4 = 10 - 12 = -2$. (As it should! :-) $\square$

b. Again, we first apply the Gauss-Jordan algorithm to $\mathbf{A}$:

$$\begin{bmatrix} 0 & 3 & 6 \\ 2 & 4 & 5 \\ 4 & 7 & 0 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\Longrightarrow} \begin{bmatrix} 2 & 4 & 5 \\ 0 & 3 & 6 \\ 4 & 7 & 0 \end{bmatrix} \overset{\frac{1}{2}R_1}{\Longrightarrow} \begin{bmatrix} 1 & 2 & \frac{5}{2} \\ 0 & 3 & 6 \\ 4 & 7 & 0 \end{bmatrix}$$

$$\underset{R_3 - 4R_1}{\Longrightarrow} \begin{bmatrix} 1 & 2 & \frac{5}{2} \\ 0 & 3 & 6 \\ 0 & -1 & -10 \end{bmatrix} \underset{\frac{1}{3}R_2}{\Longrightarrow} \begin{bmatrix} 1 & 2 & \frac{5}{2} \\ 0 & 1 & 2 \\ 0 & -1 & -10 \end{bmatrix} \underset{R_3 + R - 2}{\overset{R_1 - 2R_2}{\Longrightarrow}} \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 2 \\ 0 & 0 & -8 \end{bmatrix}$$

$$\underset{-\frac{1}{8}R_3}{\Longrightarrow} \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \overset{R_1 + \frac{3}{2}R_3,\ R_2 - 2R_3}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we check how the row operations involved changed the determinant of $\mathbf{A}$. We swapped rows once, which multiplied the determinant by -1 according to rule ii; we multiplied rows by a constant three times, each time changing the determinant by the same factor according to rule iii; and we added multiples of one row to other rows several times, which did not change the determinant by rule iv. Thus $(-1)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(-\frac{1}{8}\right)|\mathbf{A}| = |\mathbf{I}_2| = 1$, the last equality being rule i. It follows that $|\mathbf{A}| = \frac{1}{(-1)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(-\frac{1}{8}\right)} = 48$. (You can check that this agrees with the recursive definition of determinants, if you wish.) ■

2. Use rules i – v to determine $|\mathbf{A}|$ if:

a. $\mathbf{A} = \mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. *[1]*

b. $\mathbf{A}$ has a row of zeros. *[1]*

c. $\mathbf{A}$ has two equal rows. *[1]*

SOLUTIONS. **a.** A trivial row operation gives it away: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \overset{0R_1}{\Longrightarrow} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. By rule iii it follows that $|\mathbf{O}| = 0|\mathbf{O}| = 0$. $\square$

b. Same trick as in the solution to **a** above. Suppose row i of $\mathbf{A}$ is all zeros. Then $\mathbf{A} \underset{0R_i}{\Rightarrow} \mathbf{A}$, and so by rule iii it follows that $|\mathbf{A}| = 0|\mathbf{A}| = 0$. $\square$

c. Suppose rows i and j of $\mathbf{A}$ are the same. Then $\mathbf{A} \underset{R_i - R_j}{\Rightarrow} \mathbf{D}$, where *boldD*'s ith row is all zeros, and so $|\mathbf{D}| = 0$ by **b** above. By rule iv, however, subtracting one row from another does not change the determinant, so $|\mathbf{A}| = |\mathbf{D}| = 0$. ■

3. Rules ii – iv are true for the columns of $\mathbf{A}$ as well as the rows. Explain why. *[2]*

SOLUTION. The columns of $\mathbf{A}$ are the rows of $\mathbf{A}^T$, and since $|\mathbf{A}^T| = |\mathbf{A}|$ by rule v, column operations on $\mathbf{A}$ have the same effect as row operations on $\mathbf{A}^T$. It follows that rules ii – iv are true for the columns of $\mathbf{A}$. ■