Mathematics 1350H – Linear algebra I: Matrix algebra
Trent University, Summer 2014

Solutions to Assignment #4
Linear algebra for non-linear equations?

Recall that the general equation of a circle of radius r centred at the point (p, q) is $(x-p)^2 + (y-q)^2 = r^2$, and that the general equation of a parabola with a vertical axis of symmetry is $y = ax^2 + bx + c$. Consider the points $(5,1)$, $(-2,0)$, and $(6,-6)$.

1. Find the equation of the (only!) circle which pass through the three given points. *[5]*

Solution. The circle $(x-p)^2 + (y-q)^2 = r^2$ passes through the points $(5,1)$, $(-2,0)$, and $(6,-6)$ if the coordinates of each of the points satisfy the equation of the circle. The first thing to do is to plug the (x,y)-coordinates of our three points into the equation for the circle and expand:

$$\begin{aligned}
r^2 &= (5-p)^2 + (1-q)^2 = 25 - 10p + p^2 + 1 - 2q + q^2 = p^2 + q^2 - 10p - 2q + 26 \\
r^2 &= (-2-p)^2 + (0-q)^2 = 4 + 4p + p^2 + q^2 = p^2 + q^2 + 4p + 4 \\
r^2 &= (6-p)^2 + (-6-q)^2 = 36 - 12p + p^2 + 36 + 12q + q^2 = p^2 + q^2 - 12p + 12q + 72
\end{aligned}$$

It follows that $r^2 - p^2 - q^2 = -10p - 2q + 26 = 4p + 4 = -12p + 12q + 72$, and hence that $-10p - 2q + 26 = 4p + 4$, $4p + 4 = -12p + 12q + 72$, and $-12p + 12q + 72 = -10p - 2q + 26$, which we rearrange a bit to get the system of linear equations

$$\begin{aligned}
14p + 2q &= 22 \\
16p - 12q &= 68 \\
2p - 14q &= 46\,.
\end{aligned}$$

We solve this system of linear equations for p and q in the usual way:

$$\left[\begin{array}{cc|c} 14 & 2 & 22 \\ 16 & -12 & 68 \\ 2 & -14 & 46 \end{array}\right] \underset{}{\overset{R_1 \leftrightarrow R_3}{\Longrightarrow}} \left[\begin{array}{cc|c} 2 & -14 & 46 \\ 16 & -12 & 68 \\ 14 & 2 & 22 \end{array}\right] \overset{\frac{1}{2}R_1}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -7 & 23 \\ 16 & -12 & 68 \\ 14 & 2 & 22 \end{array}\right]$$

$$\underset{\substack{R_2 - 16R_1 \\ R_3 - 14R_1}}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -7 & 23 \\ 0 & 100 & -300 \\ 0 & 100 & -300 \end{array}\right] \underset{\frac{1}{100|}R_2}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -7 & 23 \\ 0 & 1 & -3 \\ 0 & 100 & -300 \end{array}\right] \underset{R_3 - 100R_2}{\overset{R_1 + 7R_2}{\Longrightarrow}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $p = 2$ and $q = -3$. We can plug this, along with the (x,y)-coordinates of one of our three points, into the equation of a circle to compute r^2:

$$r^2 = (-2-2)^2 + (0-(-3))^2 = 4^2 + 3^2 = 16 + 9 = 25$$

The circle passing through the given points thus has centre $(2,-3)$ and radius $5 = \sqrt{25}$, and its equation is $(x-2)^2 + (y+3)^2 = 25$. ∎

2. Find the equation(s) of (all) the parabola(s), if any, with a vertical axis of symmetry which pass through the three given points. *[5]*

Solution. The parabola with equation $y = ax^2 + bx + c$ passes through the points $(5, 1)$, $(-2, 0)$, and $(6, -6)$ if the coordinates of each of the points satisfy the equation. Similarly to the solution to **1** above, we plug the (x, y)-coordinates of our three points into the equation for the parabola and expand:

$$\begin{aligned} 1 &= a5^2 + b5 + c = 25a + 5b + c \\ 0 &= a(-2)^2 + b(-2) + c = 4a - 2b + c \\ -6 &= a6^2 + b6 + c = 36a + 6b + c \end{aligned}$$

This is a system of three linear equations in three unknowns, which we solve for a, b, and c in the usual way:

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & 1 \\ 4 & -2 & 1 & 0 \\ 36 & 6 & 1 & -6 \end{array}\right] \underset{\Longrightarrow}{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 4 & -2 & 1 & 0 \\ 25 & 5 & 1 & 1 \\ 36 & 6 & 1 & -6 \end{array}\right]$$

$$\underset{\Longrightarrow}{\tfrac{1}{4}R_1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 25 & 5 & 1 & 1 \\ 36 & 6 & 1 & -6 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - 25R_1 \\ R_3 - 36R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{35}{2} & -\frac{21}{4} & 1 \\ 0 & 24 & -8 & -6 \end{array}\right]$$

$$\underset{R_2 \leftrightarrow R_3}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 24 & -8 & -6 \\ 0 & \frac{35}{2} & -\frac{21}{4} & 1 \end{array}\right] \underset{\frac{1}{24}R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{4} \\ 0 & \frac{35}{2} & -\frac{21}{4} & 1 \end{array}\right]$$

$$\begin{array}{c} R_1 + \frac{1}{2}R_2 \\ \Longrightarrow \\ R_3 - \frac{35}{2}R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{12} & -\frac{1}{8} \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{4} \\ 0 & 0 & \frac{7}{12} & \frac{43}{8} \end{array}\right] \underset{\frac{12}{7}R_3}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{12} & -\frac{1}{8} \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{129}{14} \end{array}\right]$$

$$\begin{array}{c} R_1 - \frac{1}{12}R_3 \\ R_2 + \frac{1}{3}R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{25}{28} \\ 0 & 1 & 0 & \frac{79}{28} \\ 0 & 0 & 1 & \frac{129}{14} \end{array}\right]$$

Thus $a = -\frac{25}{28}$, $b = \frac{79}{28}$, and $c = \frac{129}{14}$, and so the equation of the parabola with vertical symmetry that passes through the given points is $y = -\frac{25}{28}x^2 + \frac{79}{28}x + \frac{129}{14}$. ■

Note: In general, three points in a plane that are not all in a straight line determine a unique circle that passes through all three. This can be shown, among other ways, by a souped-up version of a correct method for doing **1**.