Mathematics 1350H – Linear algebra I: Matrix algebra
TRENT UNIVERSITY, Summer 2014

Solutions to Assignment #3
Quadratic nonsense

1. Find a 2×2 matrix $\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with real entries such that $\mathbf{X}^2 + 2\mathbf{X} = -5\mathbf{I}_2$. *[5]*

Note: $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 identity matrix.

SOLUTION. We will first solve the corresponding quadratic equation for numbers and then use that to help us find a solution to the given matrix equation.

It is pretty easy to solve the corresponding equation, $x^2 + 2x = -5$, for numbers, although the answers involve complex numbers. We simply rearrange the equation into a form we can apply the quadratic formula to, $x^2 + 2x + 5 = 0$, and then apply the quadratic formula:

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4\sqrt{-1}}{2} = -1 \pm 2i\,,$$

where $i^2 = 1$. i is not a real number, to be sure, but it has its uses.

The 2×2 matrix that acts like 1 does in number systems is, of course, $\mathbf{I}_2$. To get a matrix that acts like i does for numbers, we will need a 2×2 matrix $\mathbf{J}$ such that $\mathbf{J}^2 = \mathbf{J}\mathbf{J} = -\mathbf{I}_2$. It's not too hard to find such a matrix with a little tinkering. (See the note after this solution for a more general approach.) Here are several such:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{bmatrix} \quad \begin{bmatrix} 2 & -\sqrt{5} \\ \sqrt{5} & -2 \end{bmatrix} \quad \begin{bmatrix} -\sqrt{3} & 2 \\ -2 & \sqrt{3} \end{bmatrix}$$

You can check for your self that each one of these satisfies $\mathbf{J}^2 = \mathbf{I}_2$. You may assume, if you wish, that $\mathbf{J}$ is a particular one of these in what follows, but any matrix $\mathbf{J}$ such that $\mathbf{J}^2 = \mathbf{J}\mathbf{J} = -\mathbf{I}_2$ would do as well.

The matrix analogues, $\mathbf{X} = -\mathbf{I}_2 \pm 2\mathbf{J}$, of the solutions $a = -1 \pm 2i$ to the corresponding quadratic equation for numbers are a solution to the given matrix equation. For example, let $\mathbf{X} = -\mathbf{I}_2 + 2\mathbf{J}$; then:

$$\begin{aligned} \mathbf{X}^2 + 2\mathbf{X} &= \left(-\mathbf{I}_2 + 2\mathbf{J}\right)^2 + 2\left(-\mathbf{I}_2 + 2\mathbf{J}\right) \\ &= \left(-\mathbf{I}_2\right)^2 + \left(-\mathbf{I}_2\right)\left(2\mathbf{J}\right) + \left(2\mathbf{J}\right)\left(-\mathbf{I}_2\right) + \left(2\mathbf{J}\right)^2 - 2\mathbf{I}_2 + 2\left(2\mathbf{J}\right) \\ &= \mathbf{I}_2 - 2\mathbf{J} - 2\mathbf{J} - 4\mathbf{I}_2 - 2\mathbf{I}_2 + 4\mathbf{J} = -5\mathbf{I}_2 \end{aligned}$$

A similar calculation shows that $\mathbf{X} = -\mathbf{I}_2 - 2\mathbf{J}$ is also a solution. Note that the the plethora of possible matrices $\mathbf{J}$ means that the given matrix equation has more than just two solutions. ■

NOTE. Here's a sketch of a more structured approach for finding matrices $\mathbf{J}$ such that $\mathbf{J}^2 = -\mathbf{I}_2$. If $\mathbf{J} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\mathbf{J}^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ac + bd \\ ac + bd & d^2 + bc \end{bmatrix}$.

To have $\mathbf{J}^2 = -\mathbf{I}_2$ thus requires that $a^2 + bc = -1 = d^2 + bc$ and $ac + bd = 0$. Observe first that it follows that $a^2 = d^2$, and hence that $a = \pm d$. Second, if $b = 0$ or $c = 0$ (or both), we would have to have $a^2 = d^2 = -1$, which is impossible if a and d are real numbers; it follows that we must have $b \neq 0$ and $c \neq 0$. On the other hand, if $a = 0$, and hence $d = 0$, the only thing left to satisfy would be $bc = -1$. Hence, for example, if $a = d = 0$ and $b = 1$, we would have to have $c = -1$, and $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ would be a matrix such that $\mathbf{J}^2 = -\mathbf{I}_2$. By setting some of $a - d$ to various values and trying to solve for the rest, we can get a lot of other matrices that do the job, too.

2. Is there a 2×2 matrix $\mathbf{X}$ with real entries such that $\mathbf{X}^2 + 2\mathbf{X} = -\mathbf{I}_2$, other than $\mathbf{X} = \mathbf{I}_2$? If so, find one; if not, explain why there isn't one. *[5]*

SOLUTION. First, a small sanity check: $\mathbf{I}_2$ is *not* a solution of $\mathbf{X}^2 + 2\mathbf{X} = -\mathbf{I}_2$. Now, observe that $\mathbf{X}^2 + 2\mathbf{X} = -\mathbf{I}_2 \iff \mathbf{X}^2 + 2\mathbf{X} + \mathbf{I}_2 = \mathbf{O} \iff (\mathbf{X} + \mathbf{I}_2)^2 = \mathbf{O}$. If $\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $(\mathbf{X} + \mathbf{I}_2)^2 = \begin{bmatrix} a+1 & b \\ c & d+1 \end{bmatrix}^2 = \begin{bmatrix} (a+1)^2 + bc & (a+1)b + c(d+1) \\ (a+1)b + c(d+1) & (d+1)^2 + bc \end{bmatrix}$, so we would have to have that $(a+1)^2 + bc = 0 = (d+1)^2 + bc$ and $(a+1)b + c(d+1) = ab + cd + b + c = 0$.

If, say, $b = c = 0$, then we'd have to have that $(a+1)^2 = (d+1)^2 = 0$, which would require that $a + 1 = d + 1 = 0$, and hence that $a = d = -1$. Thus $\mathbf{X} = -\mathbf{I}_2 \neq \mathbf{I}_2$ is a solution to the given equation. This would be enough to answer the question given ...

More generally, note that we must have that $(a+1)^2 = (d+1)^2$, so $a + 1 = \pm(d+1)$, and thus $a = d$ or $a = -d - 2$.

If $a = d$, $(a+1)b + c(d+1) = 0$ implies that $b + c = 0$ (or $a = d = -1$, as above), and then $(a+1)^2 + bc = 0 = (d+1)^2 + bc$ implies that $bc = -(a+1)^2 = -(d+1)^2$ as well. It is pretty easy to see that this will force $b = -c = \pm(a+1)$. For example, if we set $a = d = 1$, then $b + c = 0$ and $bc = -(1+1)^2 = -4$. Thus $a = d = 1$, $b = 2$, and $c = -2$, and so $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \neq \mathbf{I}_2$ is a solution.

If $a = -d - 2$, then we still have to have that $bc = -(a+1)^2 = -(d+1)^2$. However, $0 = (a+1)b + c(d+1) = (-d-2+1)b + c(d+1) = -(d+1)b + c(d+1) = (d+1)(-b+c)$, so either $d = -1$ (and hence $a = -(-1) - 2 = -1$, too, which would force $b = 0$ or $c = 0$) or $b = c$. But then $bc = b^2 = -(d+1)^2$, which is impossible with real numbers unless $d + 1 = 0$, *i.e.* $d = -1$ (and hence that $a = -1$, too) and $b = c = 0$. Thus this case does not give any obvious additional solutions. ■