Mathematics 1350H – Linear algebra I: Matrix algebra
TRENT UNIVERSITY, Summer 2014

Solutions to Assignment #2
Optimizing

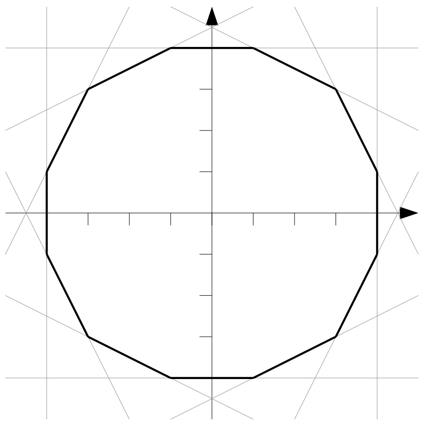
Consider the region in $\mathbb{R}^2$ consisting of all the points that satisfy *all* of the following inequalities:

$$\begin{array}{lll} -4 \leq y \leq 4 & -9 \leq 2x+y \leq 9 & -9 \leq x+2y \leq 9 \\ -4 \leq x \leq 4 & -9 \leq 2x-y \leq 9 & -9 \leq x-2y \leq 9 \end{array}$$

1. Sketch this region. *[3]*

Hint: It's a not-quite-regular dodecagon ...

SOLUTION. The idea is to draw the lines given by the corresponding inequalities, and then see what the inequalities have in commmon. Here's a sketch:



Note that $(0,0)$ satisfies every one of the given inequalities, so the region is the inside of the dodecagon, and since the inequalities are not strict, the border of the dodecagon is included. Also, the vertices of the dodecagon are the twelve points $(\pm 1, \pm 4)$, $(\pm 4, \pm 1)$, and $(\pm 3, \pm 3)$. $\square$

2. Determine the maximum value of $f(x,y) = 5x + 10y + \pi^2$ on this region. At which point(s) in the region does it occur? *[4]*

SOLUTION. Obviously, since π^2 is a constant and so does not depend on x and/or y, $f(x,y) = 5x + 10y + \pi^2$ is maximized for (x,y) in the dodecagon exactly when $5x + 10y$ is made as large as possible. Since the dodecagon is symmetric about the origin, this will occur when x and y are both positive, as otherwise one or the other part, or both, of $5x + 10y$ will be zero or negative, reducing the output. This means we really only need to consider the part of the dodecahedrom in the first quadrant, with x and y both positive.

The four inequalities that give the part of the dodecagon in the first quadrant are $y \leq 4$, $x \leq 4$, $2x + y \leq 9$, and $x + 2y \leq 9$. We'll divide up the part of the dodecagon in the first quadrant into three pieces, the rectangle $0 \leq x \leq 1$ and $0 \leq y \leq 4$, the trapezoid $1 \leq x \leq 3$ and $0 \leq y \leq \frac{9}{2} - \frac{x}{2}$ [*i.e.* the top is the line $x + 2y = 9$], and the trapezoid $3 \leq x \leq 4$ and $0 \leq y \leq 9 - 2x$ [*i.e.* the top is the line $2x + y = 9$], and then consider the maximum of $5x + 10y$ on each piece. Note that for any given value of x, $5x + 10y$ is maximized when y is made as large as possible, which means that the maximum in each of the three pieces occurs somewhere along the top edge.

First, when $0 \le x \le 1$ and $0 \le y \le 4$, it's pretty obvious that $0 \le 5x + 10y \le 5 \cdot 1 + 10 \cdot 4 = 45$ and that 45 is actually achieved at the point $(x, y) = (1, 4)$.

Second, when $1 \le x \le 3$ and $0 \le y \le \frac{9}{2} - \frac{x}{2}$, we have that along the entire top edge $5x + 10y = 5x + 10\left(\frac{9}{2} - \frac{x}{2}\right) = 5x + 45 - 5x = 45$.

Third, when $3 \le x \le 4$ and $0 \le y \le 9 - 2x$, then along the top edge $5x + 10y = 5x + 10(9 - 2x) = 5x + 90 - 20x = 90 - 15x$. Since $3 \le x \le 4$, this means that along the top edge $30 = 90 - 60 = 90 - 15 \cdot 4 \le 5x + 10y = 90 - 15x \le 90 - 15 \cdot 3 = 90 - 45 = 45$, and the maximum of 45 is achieved when $x = 3$, *i.e.* at $(x, y) = (3, 3)$.

Putting all this together, the maximum of $5x + 10y$ on the dodecagon is 45, which is achieved on (the part of the border formed by) the line $x + 2y = 9$ for $1 \le x \le 3$. It follows that the maximum of $f(x, y) = 5x + 10y + \pi^2$ on the given region is $45 + \pi^2$. □

3. Determine the minimum value of $g(x, y) = 6x - 4y + \sqrt{e}$ on this region. At which point(s) in the region does it occur? *[3]*

SOLUTION. The solution to this problem is very similar to that of **2**. We need to make $6x - 4y$ as small – that is, as negative – as possible in the dodecagon. This means that we need to look in the quadrant where x is negative and y positive, and so on. To cut to the chase, the minimum occurs exactly when $(x, y) = (-3, 3)$, and $g(-3, 3) = 6 \cdot (-3) - 4 \cdot 3 + \sqrt{e} = -30 + \sqrt{e}$. It does differ from **2** in that the minimum occurs only at a vertex instead of along an entire edge of the dodecagon. ■