Mathematics 1350H – Linear algebra I: Matrix algebra
TRENT UNIVERSITY, Summer 2014

Solutions to the Quizzes

Quiz #1. Thursday, 15 May, 2014. *[10 minutes]*

1. Find a vector in $\mathbb{R}^2$ [*i.e.* a 2-D vector] parallel to the line $y = 2x - 1$. *[2]*
2. Find a vector of length 1 parallel to the line $y = 2x - 1$. *[1]*
3. Find a vector in $\mathbb{R}^2$ perpendicular to the line $y = 2x - 1$. *[2]*

SOLUTIONS. 1. First, we find two points on the line: plugging in $x = 0$ gives $y = 2 \cdot 0 - 1 = -1$, for the point $(0, -1)$, and plugging in $x = 1$ gives $y = 2 \cdot 1 = 2 - 1 = 1$, for the point $(1, 1)$. Second, the vector that takes one from $(0, -1)$ to $(1, 1)$ is $\begin{bmatrix} 1 - 0 \\ 1 - (-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. This vector must be parallel to the given line since it runs between points on the line. □

2. The vector parallel to the given line that we obtained above has length

$$\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\| = \sqrt{1^2 + 2^2} = \sqrt{5}\,.$$

Dividing it by its own length, $\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$, gives a vector in the same direction – and so still parallel to the given line – which is of length 1. Just to be sure [overkill!], we check:

$$\left\| \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \right\| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{\frac{1}{5} + \frac{4}{5}} = \sqrt{1} = 1 \qquad \square$$

3. We need to find a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ which is perpendicular to the line. It is enough to find a vector which is perpendicular to a vector parallel to the line, such as the one obtained above in the solution to 1. Since two vector are perpendicular exactly when their dot product is 0, we therefore need to find a and b, preferably not both 0 [because the zero vector, $\mathbf{0}$, doesn't actually have a direction, if you think about it], such that

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 1a + 2b = a + 2b = 0\,.$$

This is easy enough to do by hit or miss. To be a little more systematic, set, say, $a = 1$, and solve for b:

$$1 + 2b = 0 \implies 2b = -1 \implies b = -\frac{1}{2}$$

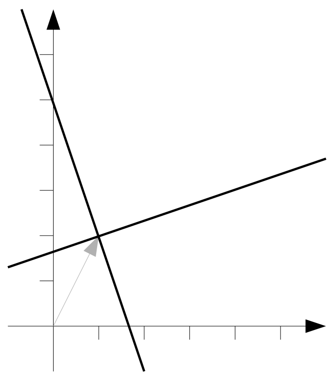
The vector $\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$ is therefore a vector which is perpendicular to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and hence to the line $y = 2x - 1$, as desired. □

Quiz #2. Tuesday, 20 May, 2014. *[12 minutes]*

Consider the lines given by the vector-parametric equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and the normal equation $\begin{bmatrix} 1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

1. Sketch both lines. *[1]*
2. What is the angle between the lines? Why? *[1]*.
3. Find equations of the form $y = mx + b$ for both lines. *[3]*

SOLUTIONS. 1. Note that the point $(1, 2)$ is on both lines, since both use the vector $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as their (initial) position vector. Also, note that the direction vector for the first line is the normal vector for the second, *i.e.* $\mathbf{d} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \mathbf{n}$. Here's a sketch of the two lines:



$\square$

2. Since the direction vector for the first line [vector parallel to the line] is the normal vector for the second [vector perpendicular to it], *i.e.* $\mathbf{d} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \mathbf{n}$, the two lines must be perpendicular to one another. $\square$

3. For the first line, we have $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1+t \\ 2-3t \end{bmatrix}$, so $x = 1+t$ and $y = 2-3t$. It follows that $t = x - 1$, and so $y = 2 - 3t = 2 - 3(x-1) = -3x + 5$.

For the second line, we simply compute the dot products in the equation,

$$x - 3y = 1x + (-3)y = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \cdot 1 + (-3) \cdot 2 = 1 - 6 = -5\,,$$

and solve for y: $x - 3y = -5 \implies 3y = x + 5 \implies y = \frac{1}{3}x + \frac{5}{3}$. $\blacksquare$

NOTE: As usual, there are lots of other ways to do these problems, especially 3.

Quiz #3. Thursday, 22 May, 2014. *[15 minutes]*

1. Find an equation of the form $ax + by + cz = d$ of the plane in $\mathbb{R}^3$ that includes the points $(-1, 2, 0)$, $(0, 2, 1)$, and $(-1, 3, 1)$. *[5]*

SOLUTION THE FIRST. We will follow the method used in last Tuesday's class:

i. Find two vectors parallel to the plane. This is easily done by using one of our three points, say $(-1, 2, 0)$, as the common base point and the other two as the tips of two vectors: $\begin{bmatrix} 0-(-1) \\ 2-2 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1-(-1) \\ 3-2 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

ii. Find a vector normal (i.e. perpendicular) to the plane. We will do this by taking the cross-product of the two vectors parallel to the plane obtained above:

$$\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k}$$

$$= (0 \cdot 1 - 1 \cdot 1)\mathbf{i} - (1 \cdot 1 - 0 \cdot 1)\mathbf{j} + (1 \cdot 1 - 0 \cdot 0)\mathbf{k} = -\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

iii. Write out the normal vector equation, $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$, of the plane. We need the position vector of a (any!) point on the plane, so we'll use the one for the point we used as a base point above, making $\mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$. Then $\mathbf{n} \cdot \mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -x - y + z$ and $\mathbf{n} \cdot \mathbf{p} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = (-1)(-1) + (-1)2 + 1 \cdot 0 = 1 - 2 + 0 = -1$, so $-x - y + z = -1$ is an equation describing the plane, as desired. (Of course, any non-zero multiple of this equation gives the same plane.) □

SOLUTION THE SECOND. We need to find a, b, c, and d (not all 0) such that $ax+by+cz = d$ is satisfied by all three of the given points:

$$\begin{array}{ccccccc} a(-1) & + & b(2) & + & c(0) & = & d \\ a(0) & + & b(2) & + & c(1) & = & d \\ a(-1) & + & b(3) & + & c(1) & = & d \end{array}, \textit{ i.e.} \text{ so } \begin{array}{ccccccccc} -a & + & 2b & & & - & d & = & 0 \\ & & 2b & + & c & - & d & = & 0 \\ -a & + & 3b & + & c & - & d & = & 0 \end{array}.$$

We will use the Gauss-Jordan method to solve the latter system of linear equations for a, b, c, and d. As usual, we will set up the augmented matrix representing the system of equations, and then reduce it using row operations:

$$\left[\begin{array}{cccc|c} -1 & 2 & 0 & -1 & 0 \\ 0 & 2 & 1 & -1 & 0 \\ -1 & 3 & 1 & -1 & 0 \end{array}\right] \underset{}{\overset{(-1)R_1}{\Longrightarrow}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 \\ -1 & 3 & 1 & -1 & 0 \end{array}\right] \underset{R_3+R_1}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\underset{R_2 \leftrightarrow R_3}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 0 \end{array}\right] \underset{R_3-2R_2}{\overset{R_1+2R_2}{\Longrightarrow}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right]$$

$$\underset{(-1)R_3}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \overset{R_1-2R_3,\ R_2-R_3}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

The last matrix represents the much-simplified system of equations $a - d = 0$, $b - d = 0$, and $c + d = 0$, *i.e.* $a = d$, $b = d$, and $c = -d$. d can be anything we like (though we do need it to be non-zero to get something useful), so we'll set to 1, so $a = b = d = 1$ and $c = -1$. This means that one possible equation for the plane is $x + y - z = 1$. Note that this is just the multiple by -1 of the equation obtained in the previous solution. ■

Quiz #4. Tuesday, 27 May, 2014. *[15 minutes]*

1. Find all the solutions, if any, to the following system of linear equations:

$$\begin{array}{rcrcrcr} x & + & y & & & = & 2 \\ 2x & + & y & - & z & = & 2 \\ x & + & 2y & - & z & = & 0 \end{array}$$

[4]

2. Each of the equations in the given system represents a plane in $\mathbb{R}^3$. What does your answer to the question above tell you about how these planes intersect? *[1]*

Solutions. 1. We set up the augmented matrix corresponding to the given system of equations and run through the Gauss-Jordan method:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 2 & 1 & -1 & 2 \\ 1 & 2 & -1 & 0 \end{array}\right] \underset{R_3-R_1}{\overset{R_2-2R_1}{\Longrightarrow}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & 1 & -1 & -2 \end{array}\right] \underset{(-1)R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$$\underset{R_3-R_2}{\overset{R_1-R_1}{\Longrightarrow}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -4 \end{array}\right] \underset{-\frac{1}{2}R_3}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \overset{R_1+R_3,\ R_2-R_3}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Thus $x = 2$, $y = 0$, and $z = 2$ is the only solution to the given system of equations. □

2. Since $x = 2$, $y = 0$, and $z = 2$ is the only solution to the given system of linear equations, the three planes intersect in the single point $(2, 0, 2)$. ■

Quiz #5. Thursday, 29 May, 2014. *[15 minutes]*

1. Determine whether $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \in \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ or not. *[4]*

2. What, if anything, does your answer to question 1 tell you about whether or not $\left\{ \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a linearly independent set of vectors? *[1]*

SOLUTION. 1. By definition, this boils down to asking whether there are real numbers a, b, and c such that $a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$, which boils down to asking whether the system of equations $\begin{array}{ccccccc} a & + & b & + & 3c & = & 5 \\ a & + & 2b & + & 2c & = & 3 \\ a & + & 3b & + & c & = & 1 \end{array}$ has a solution. We check this by setting up the corresponding augmented matrix and applying the Gauss-Jordan method:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 1 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & -2 & -4 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \Longrightarrow \\ R_3 - 2R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The final matrix corresponds to the reduced system of equations $\begin{array}{ccccc} a & + & 4c & = & 7 \\ b & - & c & = & -2 \end{array}$, which has infinitely many solutions: c can be set to anything you like and then a and b solved for. For example, if $c = 0$, then $a = 7$ and $b = -2$. Since there are solutions, it follows that $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \in \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$. □

2. Since $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \in \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$, $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$ is a linear combination of the other three vectors, and so all four vectors taken together must be linearly dependent instead of independent. For example, since $7 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$, it follows that $(-1) \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} + 7 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since not all of these scalars are 0, the four vectors are linearly dependent. ■

Quiz #6. Thursday, 5 June, 2014. *[15 minutes]*

1. Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices such that $\mathbf{A}$ and $\mathbf{AB}$ both have inverses. Either show that $\mathbf{B}$ must have an inverse too, or give an example demonstrating that $\mathbf{B}$ does not have to have an inverse. *[5]*

SOLUTION. If $\mathbf{A}$ and $\mathbf{AB}$ both have inverses, then so does $\mathbf{B}$. The easiest way to see this is probably to write the inverse of $\mathbf{B}$ in terms of $\mathbf{A}$ and $\mathbf{AB}$. The easiest way to do that, in turn, is to assume that $\mathbf{B}$ is invertible, write the inverse of $\mathbf{AB}$ in terms of the inverses of $\mathbf{A}$ and $\mathbf{B}$, and then solve for $\mathbf{B}^{-1}$:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \implies \mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{A} = (\mathbf{AB})^{-1}\mathbf{A}$$

The matrix $(\mathbf{AB})^{-1}\mathbf{A}$, which must exist, given that we assumed that $\mathbf{AB}$ has an inverse, therefore ought to be the inverse of $\mathbf{B}$. Just to be sure, we check:

$$\left[(\mathbf{AB})^{-1}\mathbf{A}\right]\mathbf{B} = (\mathbf{AB})^{-1}(\mathbf{AB}) = \mathbf{I}_n$$

Thus $(\mathbf{AB})^{-1}\mathbf{A}$ is the inverse matrix for $\mathbf{B}$, so it has one. ■

Quiz #7. Tuesday, 10 June, 2014. *[15 minutes]*

Determine whether each of the following collections of vectors is a subspace of $\mathbb{R}^2$.

1. $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x + y = 1 \right\}$ *[1.5]*
2. $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x + y = 0 \right\}$ *[1.5]*
3. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x + y = 0 \text{ and/or } x - y = 0 \right\}$ *[2]*

SOLUTIONS. 1. U is not a subspace of $\mathbb{R}^2$, because $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin \mathbb{R}^2$, as $0 + 0 = 0 \neq 1$. (Recall from class, or the textbook, that the zero vector must be in every subspace.) □

2. V is a subspace of $\mathbb{R}^2$. Suppose $\begin{bmatrix} s \\ t \end{bmatrix}$ and $\begin{bmatrix} u \\ v \end{bmatrix}$ are in V, so $s + t = 0 = u + v$, and that a and b are any scalars. Then $a\begin{bmatrix} s \\ t \end{bmatrix} + b\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} as + bu \\ at + bv \end{bmatrix}$, for which $(as + bu) + (at + bv) = a(s + t) + b(u + v) = a \cdot 0 + b \cdot 0 = 0 + 0 = 0$. By the definition of U, this means that $a\begin{bmatrix} s \\ t \end{bmatrix} + b\begin{bmatrix} u \\ v \end{bmatrix}$ is in U, too. Since it satisfies (one the versions of) the definition of a subspace, U is a subspace of $\mathbb{R}^2$. □

3. W is not a subspace of $\mathbb{R}^2$. To see this, note that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} inW$ because $1 - 1 = 0$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix} inW$ because $1 + (-1) = 0$, but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin W$ because $2 + 0 = 2 \neq 0$ and $2 - 0 = 2 \neq 0$. Since it fails one of the conditions of the definition of a subspace, W is not a subspace of $\mathbb{R}^2$. ■

Quiz #8. *Take-home! [Due in class on Tuesday, 17 June, 2014.]*

You may use your textbook, your notes, and all handouts from this class, but you may not consult any other sources or persons.

1. Mr. Pisistratus Patriarch lived up to his somewhat unusual name. He had nine children, and no fewer than 31 grandchildren. In his will he left an exact number of dollars to each grandchild. Each girl was to get \$7 more than each boy. All 31 grandchildren were alive when Patriarch died, and their legacies totaled \$470. Of this amount, \$74 went to Mrs. Inkpen's children (she was Patriarch's eldest daughter). How many daughters had Mrs. Inkpen? Please give your reasoning in detail. *[5]*

SOLUTION. If Mr. Patriarch had f granddaughters and m grandsons, then we know that $f + m = 31$. Also, if each grandson received a legacy of $\$\ell$, then each granddaughter received a legacy of $\$\ell + 7$, so $m\ell + f(\ell + 7) = 470$. We may reasonably assume that f, m, and ℓ are all integers which are not negative.

Solving for f in the first equation, $f = 31 - m$, and substituting this into the second gives

$$\begin{aligned} 470 &= m\ell + (31 - m)(\ell + 7) \\ &= m\ell + 31\ell + 217 - m\ell - 7m \\ &= 31\ell - 7m + 217\,, \end{aligned}$$

which simplifies down to

$$31\ell - 7m = 253\,.$$

There is only one way to make this equation hold if m is a non-negative integer ≤ 31, namely $m = 17$ and $\ell = 12$. (Just try every m between 0 and 31 ...) Thus Mr. Patriarch had 17 grandsons, each of whom received \$12, and $14 = 31 - 17$ granddaughters, each of whom received $\$19 = 12 + 7$.

Now suppose that Mrs. Inkpen had d daughters and s sons. Since her children received \$74 between them, we have that $19d + 12s = 74$. The only way to make this equation hold if d and s are non-negative integers is to make $d = 2$ and $s = 3$. (Brute force again ...) Thus Mrs. Inkpen had two daughters. ■

NOTE: This puzzle was created by one Hubert Phillips, using the pen-name "Caliban".

Quiz #9. Tuesday, 17 June, 2014. *[15 minutes]*

Let $\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$.

1. Use the Gauss-Jordan method to row-reduce $\mathbf{A}$. *[2]*
2. Find a basis for $\text{col}(\mathbf{A})$. *[1]*
3. Find a basis for $\text{row}(\mathbf{A})$. *[1]*
4. Find a basis for $\text{null}(\mathbf{A})$. *[1]*

SOLUTIONS. 1. Here goes:

$$\begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix} \begin{matrix} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 4 & 7 & 10 \\ 0 & -3 & -6 & -9 \\ 0 & -6 & -12 & -18 \end{bmatrix}$$

$$\begin{matrix} \Longrightarrow \\ -\frac{1}{3}R_2 \\ \end{matrix} \begin{bmatrix} 1 & 4 & 7 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & -6 & -12 & -18 \end{bmatrix} \begin{matrix} R_1 - 4R_2 \\ \Longrightarrow \\ R_3 + 6R_2 \end{matrix} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \square$$

2. The leading 1s of non-zero rows of the reduced matrix are in columns 1 and 2, so the corresponding columns of the original matrix are a basis for the column space, *i.e.* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ is a basis for $\text{col}(\mathbf{A})$. $\square$

3. The non-zero rows of the reduced matrix form a basis of the rown space; writing them as columns, $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis for $\text{row}(\mathbf{A})$. $\square$

4. Recall that $\text{null}(\mathbf{A}) = \{ \mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{0} \}$, that is, it consists of all the solutions of the homogeneous system of equations whose coefficient matrix is $\mathbf{A}$. If $\mathbf{x} = [x \quad y \quad z \quad w]^T$, then the reduced version of the homogenous system is the pair of equations $x - z - 2w = 0$ and $y + 2z + 3w = 0$. Note that we can solve for $x = z + 2w$ and $y = -2z - 3w$ for any values of z and w. Thus a vector-parametric description of the set of solutions, using the parameters s and t, would be:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

It follows that $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for $\text{null}(\mathbf{A})$. $\blacksquare$

Quiz #10. Thursday, 19 June, 2014. *[15 minutes]*

1. Find the eigenvalues of $\mathbf{A} = \begin{bmatrix} 4 & 4 \\ 5 & 3 \end{bmatrix}$. *[3]*
2. Find a non-zero eigenvector for each eigenvalue of $\mathbf{A}$. *[2]*

SOLUTIONS. 1. To find the eigenvalues, we first work out $|\mathbf{A} - \lambda \mathbf{I}_2|$:

$$|\mathbf{A} - \lambda \mathbf{I}_2| = \left| \begin{bmatrix} 4 & 4 \\ 5 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 4-\lambda & 4 \\ 5 & 3-\lambda \end{vmatrix}$$
$$= (4-\lambda)(3-\lambda) - 4 \cdot 5 = 12 - 7\lambda + \lambda^2 - 20 = \lambda^2 - 7\lambda - 8$$

The eigenvalues are the roots of this polynomial; setting $\lambda^2 - 7\lambda - 8 = 0$ and applying the quadratic formula gives:

$$\lambda = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 1 \cdot (-8)}}{2 \cdot 1} = \frac{7 \pm \sqrt{49+32}}{2} = \frac{7 \pm \sqrt{81}}{2} = \frac{7 \pm 9}{2} = +8 \text{ or } -1$$

Thus the eigenvalues of $\mathbf{A}$ are $+8$ and -1. □

2. For each of $\lambda = 8$ and -1, we need to find a vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. This is equivalent to finding a vector $\mathbf{x} \neq \mathbf{0}$ such that $(\mathbf{A} - \lambda \mathbf{I}_2)\,\mathbf{x} = \mathbf{0}$.

If $\lambda = 8$, $\mathbf{A} - \lambda \mathbf{I}_2 = \mathbf{A} - 8\mathbf{I}_2 = \begin{bmatrix} 4-8 & 4 \\ 5 & 3-8 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 5 & -5 \end{bmatrix}$. We need to find $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ such that $\begin{bmatrix} -4 & 4 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, *i.e.* such that $\left.\begin{matrix} -4x + 4y = 0 \\ 5x - 5y = 0 \end{matrix}\right\} \Rightarrow y = x$. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will do.

If $\lambda = -1$, $\mathbf{A} - \lambda \mathbf{I}_2 = \mathbf{A} - (-1)\mathbf{I}_2 = \begin{bmatrix} 4+1 & 4 \\ 5 & 3+1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 5 & 4 \end{bmatrix}$. We need to find $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ such that $\begin{bmatrix} 5 & 4 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, *i.e.* such that $\left.\begin{matrix} 5x + 4y = 0 \\ 5x + 4y = 0 \end{matrix}\right\} \Rightarrow y = -\frac{5}{4}x$. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$ will do. ■