Mathematics 1350H – Linear algebra I: Matrix algebra
TRENT UNIVERSITY, Summer 2014

FINAL EXAMINATION
Friday, 20 June, 2014

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Do parts **I** and **II**, and, if you wish, part **III**. Show all your work. *If in doubt about something,* **ask!**

Aids: Calculator; one 8.5″ × 11″ or A4 aid sheet; no neuron limit.

Part I. Do *all four* (4) of **1**–**4**. *[Subtotal = 64/100]*

1. Consider the following system of linear equations:

$$\begin{array}{rcrcrcrcrcrcl} 2u & & & + & 4x & & & + & 6z & = & 0 \\ & & v & & & + & y & & & = & 0 \\ u & - & v & + & x & - & y & + & z & = & 0 \\ 3u & & & + & 5x & & & + & 7z & = & 0 \end{array}$$

a. With as little work as possible (but which you should fully show!), determine whether this system has no solution, just one solution, or many solutions. *[6]*

b. Use your answer to **a** to determine if $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 1 \\ 7 \end{bmatrix} \right\}$ is a linearly dependent or independent set of vectors. *[4]*

2. Consider the planes in $\mathbb{R}^3$ given by the equations $x + y + z = 4$ and $x + 2y + 3z = 6$.

a. Find a parametric description of the line in which the two planes intersect. *[5]*

b. Find a vector parallel to both planes. *[5[*

c. Sketch the two planes and the line. *[5]*

3. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

a. Compute $\mathbf{A}^{-1}$, if it exists. *[10]*

b. Use your calculation for **a** to find $|\mathbf{A}|$. *[5]*

c. What are the rank and nullity of $\mathbf{A}$? Why? *[2]*

d. Compute $\mathbf{A}^T$. *[2]*

4. Consider the subspace $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ of $\mathbb{R}^4$.

Do **a** and *one* (1) of **b** or **c**.

a. Find a basis for W. *[10]*

b. Find an orthogonal basis for W. *[10]*

c. Find a matrix $\mathbf{B}$ such that $W = \{ \mathbf{x} \in \mathbb{R}^4 \mid \mathbf{B}\mathbf{x} = \mathbf{0} \}$. *[10]*

Part II. Do any *three* (3) of **6–11**. *[Subtotal = 36/100]*

6. Find all the eigenvalues of $\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, and a nonzero eigenvector for each eigenvalue. *[12]*

7. Determine whether each of the following is a subspace of $\mathbb{R}^2$ or not. *[12 = 3 × 4 each]*

a. $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; 2x = 3y \right\}$

b. $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; (x+1)^2 = (x-1)^2 \right\}$

c. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; |x - y| = 0 \right\}$

8. Find an example of 2×2 matrices $\mathbf{X}$ and $\mathbf{Y}$ satisfying the equations $\mathbf{X} + \mathbf{Y} = \mathbf{I}_2$ and $\mathbf{X} - 2\mathbf{Y} = \mathbf{O}_2$, where $\mathbf{I}_2$ and $\mathbf{O}_2$ are the 2×2 identity and zero matrices, respectively. How many such matrices $\mathbf{X}$ and $\mathbf{Y}$ are there? Explain why. *[12]*

9. **a.** Suppose $\mathbf{w} \in \mathbb{R}^n$ is perpendicular to all of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k \in \mathbb{R}^n$. Show that if $\mathbf{u} \in \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k\}$, then $\mathbf{w}$ and $\mathbf{u}$ are also perpendicular. *[8]*

b. How large can a collection of vectors can one find in $\mathbb{R}^n$ such that each is perpendicular to every other vector in the collection? Why? *[4]*

10. Suppose the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find the matrix $[T]$ such that $T(\mathbf{x}) = [T]\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$. *[12]*

11. Use the Gauss-Jordan method to find all the solutions, if any, of the system of equations given in question **1**. *[12]*

[Total = 100]

Part III. Bonus!

0. Write an original little poem about linear algebra or mathematics in general. *[1]*

-5. Give a creative explanation for the lack of a question **5** on this exam. *[1]*

ENJOY THE REST OF THE SUMMER!