Mathematics 1350H – Linear algebra I: Matrix algebra
TRENT UNIVERSITY, Summer 2014

SOLUTIONS TO THE FINAL EXAMINATION
Friday, 20 June, 2014

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Do parts **I** and **II**, and, if you wish, part **III**. Show all your work. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; no neuron limit.

Part I. Do *all four* (4) of **1**–**4**. *[Subtotal = 64/100]*

1. Consider the following system of linear equations:

$$\begin{array}{rcrcrcrcrcl} 2u & & & + & 4x & & & + & 6z & = & 0 \\ & & v & & & + & y & & & = & 0 \\ u & - & v & + & x & - & y & + & z & = & 0 \\ 3u & & & + & 5x & & & + & 7z & = & 0 \end{array}$$

a. With as little work as possible (but which you should fully show!), determine whether this system has no solution, just one solution, or many solutions. *[6]*

b. Use your answer to **a** to determine if $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 1 \\ 7 \end{bmatrix} \right\}$ is a linearly dependent or independent set of vectors. *[4]*

SOLUTIONS. **a.** This is a homogeneous system of linear equations, so it must have at least one solution, namely $u = v = x = y = z = 0$. Since there are five variables but only four equations in the system, it cannot have exactly one solution. Thus the given system must have many solutions. □

b. Each of the five vectors consists of the coefficients of one of the five variables in the given sistem of linear equations, so each solution to the system gives the coefficients of a linear combination of the vectors adding up to the zero vector. Since there are many such solutions, there are linear combinations of the five vectors with not all coefficients equal to zero making the linear combination of these vectors equal to zero. By definition, this means that the vectors are linearly dependent. ■

2. Consider the planes in $\mathbb{R}^3$ given by the equations $x + y + z = 4$ and $x + 2y + 3z = 6$.

a. Find a parametric description of the line in which the two planes intersect. *[5]*

b. Find a vector parallel to both planes. *[5[*

c. Sketch the two planes and the line. *[5]*

SOLUTIONS. **a.** The line consists of the common solutions to the equations of the planes, so we set up the appropriate augmented matrix and apply the Gauss-Jordan algorithm:

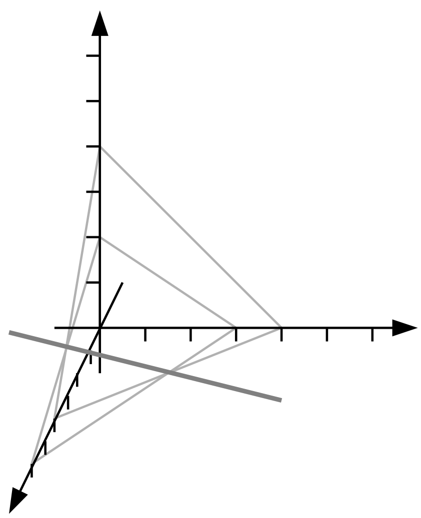
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 6 \end{array}\right] \underset{R_2 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 2 \end{array}\right] \overset{R_1 - R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

That is, we have $x - z = 2$ and $y + 2z = 2$ on the line in question. Setting $z = t$ for a parameter $t \in \mathbb{R}$ gives $x = 2 + z = 2 + t$ and $y = 2 - 2z = 2 - 2t$, so a vector-parametric descrition of the line would be:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+t \\ 2-2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \square$$

b. A direction vector of the line in which the two planes intersect must be parallel to both planes. It follows from the solution to **a** that $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a vector parallel to both planes. $\square$

c. Here is a crude sketch of the planes and their line of intersection:



$\blacksquare$

3. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

a. Compute $\mathbf{A}^{-1}$, if it exists. *[10]*
b. Use your calculation for **a** to find $|\mathbf{A}|$. *[5]*
c. What are the rank and nullity of $\mathbf{A}$? Why? *[2]*
d. Compute $\mathbf{A}^T$. *[2]*

SOLUTIONS. **a.** As usual, we set up the "super-augmented" matrix and do the Gauss-Jordan:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_2 - R_1 \\ R_3 - R_1 \\ \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \Longrightarrow \\ R_3 - R_2 \\ R_4 - R_2 \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} R_1 - R_3 \\ R_2 + R_3 \\ \Longrightarrow \\ R_4 - 2R_3 \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \Longrightarrow \\ \\ \frac{1}{3}R_4 \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right]$$

$$\begin{array}{r} R_1 - 2R_4 \\ R_2 + R_4 \\ R_3 + R_4 \\ \Longrightarrow \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right]$$

Thus $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$. $\square$

b. Looking only at the left part of the augmented matrices in the calculation above, note that adding or subtracting one row from another does not change the determinant of a matrix, and no row swaps were performed, so the only operation that changed the determinant was the multiplication of row 4 by $\frac{1}{3}$ at the next-to-last step. Multiplying a row by a constant multiplies the determinant by the same constant, so $|\mathbf{I}| = \frac{1}{3}|\mathbf{A}$. It follows that $|\mathbf{A}| = 3|\mathbf{I}| = 3 \cdot 1 = 3$. $\square$

c. The rank of $\mathbf{A}$ is the number of non-zero rows remaining after it has been fully reduced. Since there are 4 non-zero rows on the left-hand side in the solution to part **a**, $\text{rank}(\mathbf{A}) = 4$. It follows that the nullity of $\mathbf{A}$, which is the number of columns of $\mathbf{A}$ minus the rank of $\mathbf{A}$, is $4 - 4 = 0$. $\square$

d. To compute the transpose of a matrix, we simply interchange rows with columns:

$$\mathbf{A}^T = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad \blacksquare$$

4. Consider the subspace $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ of $\mathbb{R}^4$.

Do **a** and *one* (1) of **b** or **c**.

a. Find a basis for W. *[10]*

b. Find an orthogonal basis for W. *[10]*

c. Find a matrix $\mathbf{B}$ such that $W = \left\{ \mathbf{x} \in \mathbb{R}^4 \mid \mathbf{Bx} = \mathbf{0} \right\}$. *[10]*

Solutions. **a.** We assemble the vectors in the spanning set into the columns of a matrix and apply the Gauss-Jordan algorithm:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \underset{R_3 - R_1}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \underset{\Longrightarrow}{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{matrix} R_1 - R_2 \\ \Longrightarrow \\ R_3 + R_2 \end{matrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 + R_3 \\ R_2 - R_3 \\ \Longrightarrow \\ R_4 - R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The vectors which are the columns of the original matrix that correspond to the columns of the reduced matrix in which leading 1s of rows occur form a basis for W. It follows that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for W. $\square$

b. We will modify the basis obtained in the solution to **a** to make an orthogonal basis. For convenience, let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$; the orthogonal basis vectors will be denoted by $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$. Applying the Gram-Schmidt orthogonalization process gives us:

$$\mathbf{b}_1 = \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \mathbf{b}_2 &= \mathbf{u}_2 - \text{proj}_{\mathbf{b}_1}(\mathbf{u}_2) = \mathbf{u}_2 - \frac{\mathbf{b}_1 \cdot \mathbf{u}_2}{\mathbf{b}_1 \cdot \mathbf{b}_1} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1}{1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \end{aligned}$$

$$\mathbf{b}_3 = \mathbf{u}_3 - \text{proj}_{\mathbf{b}_1}(\mathbf{u}_3) - \text{proj}_{\mathbf{b}_2}(\mathbf{u}_3) = \mathbf{u}_2 - \frac{\mathbf{b}_1 \cdot \mathbf{u}_3}{\mathbf{b}_1 \cdot \mathbf{b}_1} - \frac{\mathbf{b}_2 \cdot \mathbf{u}_3}{\mathbf{b}_2 \cdot \mathbf{b}_2}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1}{1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{\frac{1}{2} \cdot 0 + 0 \cdot 1 + \left(-\frac{1}{2}\right) \cdot 0 + 1 \cdot 1}{\frac{1}{2} \cdot \frac{1}{2} + 0 \cdot 0 + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + 1 \cdot 1} \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{0}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3/2} \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ 0 \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 1 \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

Thus $\mathcal{O} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ 1 \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \right\}$ is an orthogonal nasis for the subspace W. (It's easy to check, if you're so inclined, that the vectors in the modified basis are mutually orthogonal.) □

c. At first glance, this problem seems unlike most of the problems found in the text or done in class, but there is a little less to it than one might first guess. We need to find a matrix $\mathbf{B}$ whose null space is precisely the subspace W of $\mathbb{R}^4$, which requires $\mathbf{B}$ to have four columns. Since, by the solution to **a**, W has dimension three, $\mathbf{B}$ must have rank equal to the number of its columns minus its nullity, *i.e.* $4 - 3 = 1$. It follows that $\mathbf{B}$ really only needs one row. Moreover, we only need to ensure that $\mathbf{Bx} = \mathbf{0}$ only when $\mathbf{x}$ is an element of some particular basis for W. (Because every other vector in W is a linear combination of basis elements and vector multiplication distributes over vector addition ...)

Putting all of the above together cuts our problem down to finding a matrix $\mathbf{B} = [a \quad b \quad c \quad d]$ such that, using the notation from the solution to part **b** above, $\mathbf{Bu}_1 = \mathbf{0}$, $\mathbf{Bu}_2 = \mathbf{0}$, and $\mathbf{Bu}_3 = \mathbf{0}$. Writing these out gives us

$$\begin{array}{rcl} a + c & = & 0 \\ a + d & = & 0 \\ b + d & = & 0 \end{array}$$

... which is a system of linear equations we need to solve. We set up the corresponding augmented matrix and apply the Gauss-Jordan method:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \underset{R_2 - R_1}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \underset{R_2 \leftrightarrow R_3}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

$$\underset{(-1)R_3}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \overset{R_1 - R_3}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

That is, $a + c = 0$, $b + d = 0$, & $c - d = 0$.

It is clear that we can set d to any value and solve for the other entries of $\mathbf{B}$. We arbitrarily set $d = 1$, then $c = 1$, $b = -1$ and $a = -1$. Thus $\mathbf{B} = [-1 \quad -1 \quad 1 \quad 1]$ has W as its null space, as required. ■

Part II. Do any *three* (3) of **6–11**. *[Subtotal = 36/100]*

6. Find all the eigenvalues of $\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, and a nonzero eigenvector for each eigenvalue. *[12]*

SOLUTION. We first compute and factor $|\mathbf{B} - \lambda\mathbf{I}|$:

$$|\mathbf{B} - \lambda\mathbf{I}| = \left| \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix}$$

We expand the determinant along the first row.

$$\begin{aligned} &= (1-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 1 & 1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 1-\lambda \\ 1 & 0 \end{vmatrix} \\ &= (1-\lambda)\left[(1-\lambda)(1-\lambda) - 0 \cdot 0\right] - 0 + 1\left[0 \cdot 0 - 1(1-\lambda)\right] \\ &= (1-\lambda)^3 - (1-\lambda) = (1-\lambda)\left[(1-\lambda)^2 - 1\right] \\ &= (1-\lambda)\left[1 - 2\lambda + \lambda^2 - 1\right] = (1-\lambda)\lambda(\lambda-2) \end{aligned}$$

It follows that $|\mathbf{B}-\lambda\mathbf{I}| = (1-\lambda)\lambda(\lambda-2) = 0$ exactly when $\lambda = 0$, 1, or 2, so the eigenvalues of $\mathbf{B}$ are 0, 1, and 2.

It remains to find a non-zero eigenvector for each eigenvalue; this boils down to finding non-zero solutions to the system of linear equations $(\mathbf{B} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for each of $\lambda = 0$, 1, and 2. We set up the augmented matrix in each case and Gauss-Jordan away:

$$[\mathbf{B} - 0\mathbf{I}|\,\mathbf{0}] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \underset{R_3 - R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \text{So for } \lambda = 0, \\ x + z = 0 \text{ and} \\ y = 0. \end{array}$$

$$[\mathbf{B} - 1\mathbf{I}|\,\mathbf{0}] = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 \to R_2 \\ \to R_3 \to R_1 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \text{So for } \lambda = 1, \\ x = z = 0 \text{ and} \\ y = \text{anything}. \end{array}$$

$$[\mathbf{B} - 2\mathbf{I}|\,\mathbf{0}] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_3 \\ (-1)R_2 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{R_3 + R_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \text{So for } \lambda = 2, \\ x - z = 0 \text{ and} \\ y = 0. \end{array}$$

To get the vectors we need for each eigenvalue, we set one of the variables we can control to 1 and solve for the others as necessary:

$$\lambda = 0 : \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \lambda = 1 : \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda = 2 : \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \blacksquare$$

7. Determine whether each of the following is a subspace of $\mathbb{R}^2$ or not. *[12 = 3 × 4 each]*

a. $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; 2x = 3y \right\}$

b. $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; (x+1)^2 = (x-1)^2 \right\}$

c. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; |x-y| = 0 \right\}$

SOLUTIONS. **a.** Suppose $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ are in U, so $2a = 3b$ and $2c = 3d$, and s is a scalar. Then $2(a+c) = 2a + 2c = 3b + 3d = 3(b+d)$, so $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$ is also in U, and $2(sa) = s(2a) = s(3b) = 3(sb)$, so $s\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} sa \\ sb \end{bmatrix}$ is also in U. Since U is closed under vector addition and multiplication by scalars, U is a subspace of $\mathbb{R}^2$. □

b. First, note that $x^2 + 2x + 1 = (x+1)^2 = (x-1)^2 = x^2 - 2x + 1$ is true if and only if $2x = -2x$, *i.e.* if and only if $x = 0$. Suppose $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ are in V, so $a = 0$ and $c = 0$, and s is a scalar. Then $2(a+c) = 2 \cdot 0 = 0$, so $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$ is also in V, and $2(sa) = s(2 \cdot 0) = s \cdot 0 = 0$, so $s\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} sa \\ sb \end{bmatrix}$ is also in V. Since V is closed under vector addition and multiplication by scalars, V is a subspace of $\mathbb{R}^2$. □

c. First, note that $|x-y| = 0$ is true if and only if $x - y = 0$, *i.e.* if and only if $x = y$. Suppose $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ are in V, so $a = b$ and $c = d$, and s is a scalar. Then $2(a+c) = 2a + 2c = 2b + 2d = 2(b+d)$, so $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$ is also in W, and $2(sa) = s(2a) = s(2b) = 2(sb)$, so $s\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} sa \\ sb \end{bmatrix}$ is also in W. Since W is closed under vector addition and multiplication by scalars, W is a subspace of $\mathbb{R}^2$. ■

8. Find an example of 2×2 matrices $\mathbf{X}$ and $\mathbf{Y}$ satisfying the equations $\mathbf{X} + \mathbf{Y} = \mathbf{I}_2$ and $\mathbf{X} - 2\mathbf{Y} = \mathbf{O}_2$, where $\mathbf{I}_2$ and $\mathbf{O}_2$ are the 2×2 identity and zero matrices, respectively. How many such matrices $\mathbf{X}$ and $\mathbf{Y}$ are there? Explain why. *[12]*

SOLUTION. There is only one pair of matrices satisfying both equations. First, $\mathbf{X} - 2\mathbf{Y} = \mathbf{O}_2$ implies that $\mathbf{X} = \mathbf{O}_2 + 2\mathbf{Y} = 2\mathbf{Y}$. Second, substituting this into the other equation gives us $\mathbf{I}_2 = \mathbf{X} + \mathbf{Y} = 2\mathbf{Y} + \mathbf{Y} = 3\mathbf{Y}$. It follows that $\mathbf{Y} = \frac{1}{3}\mathbf{I}_2 = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ and

$$\mathbf{X} = 2\mathbf{Y} = \tfrac{2}{3}\mathbf{I}_2 = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}. \ ■$$

9. **a.** Suppose $\mathbf{w} \in \mathbb{R}^n$ is perpendicular to all of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k \in \mathbb{R}^n$. Show that if $\mathbf{u} \in \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$, then $\mathbf{w}$ and $\mathbf{u}$ are also perpendicular. *[8]*

b. How large can a collection of vectors can one find in $\mathbb{R}^n$ such that each is perpendicular to every other vector in the collection? Why? *[4]*

SOLUTIONS. **a.** Recall that $\mathbf{w} \in \mathbb{R}^n$ is perpendicular to $\mathbf{b}$ exactly when $\mathbf{w} \cdot \mathbf{b} = 0$. Suppose $\mathbf{u} \in \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$; then $\mathbf{u} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_k\mathbf{a}_k$ for some scalars $c_1, c_2, \ldots, c_k$. Then

$$\begin{aligned}\mathbf{w} \cdot \mathbf{u} &= \mathbf{w} \cdot (c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_k\mathbf{a}_k) = \mathbf{w} \cdot (c_1\mathbf{a}_1) + \mathbf{w} \cdot (c_2\mathbf{a}_2) + \cdots + \mathbf{w} \cdot (c_k\mathbf{a}_k) \\ &= c_1(\mathbf{w} \cdot \mathbf{a}_1) + c_2(\mathbf{w} \cdot \mathbf{a}_2) + \cdots + c_k(\mathbf{w} \cdot \mathbf{a}_k) = c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_k \cdot 0 = 0,\end{aligned}$$

so $\mathbf{w}$ and $\mathbf{u}$ are perpendicular. □

b. You can find at most n mutually perpendicular vectors in $\mathbb{R}^n$. For example, the standard basis, $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$, of $\mathbb{R}^n$ is such a collection of vectors. One cannot find a larger collection of mutually perpendicular vectors in $\mathbb{R}^n$ because any collection of mutually perpendicular vectors must be linearly independent (think about it!), and you can have at most n linearly independent vectors in an n-dimensional space. ■

10. Suppose the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$ and $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$. Find the matrix $[T]$ such that $T(\mathbf{x}) = [T]\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$. *[12]*

SOLUTION. The key here is to observe that $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$ because the two vectors are not multiples of one another, and hence are linearly independent. (Any two linearly independent vectors in (sub)space of dimension 2 form a basis ...) Since T takes this basis to the standard basis of $\mathbb{R}^2$, T must be invertible. It turns out to be easy to find the matrix of T^{-1}; inverting this matrix will give us the matrix of T.

First, given what we are told about T, we must have $T^{-1}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$ and $T^{-1}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$. Suppose $[T^{-1}] = \begin{bmatrix}a & c\\ b & d\end{bmatrix}$. Then

$$\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}a & c\\ b & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = [T^{-1}]\begin{bmatrix}1\\0\end{bmatrix} = T^{-1}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}a & c\\ b & d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = [T^{-1}]\begin{bmatrix}0\\1\end{bmatrix} = T^{-1}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix},$$

so $[T^{-1}] = \begin{bmatrix}a & c\\ b & d\end{bmatrix} = \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$. Second, we invert this matrix in the usual way:

$$\left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\ 2 & 1 & 0 & 1\end{array}\right] \underset{R_2 - 2R_1}{\Longrightarrow} \left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\ 0 & -3 & -2 & 1\end{array}\right]$$

$$\underset{-\frac{1}{3}R_2}{\Longrightarrow} \left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\ 0 & 1 & \frac{2}{3} & -\frac{1}{3}\end{array}\right] \overset{R_1 - 2R_2}{\Longrightarrow} \left[\begin{array}{cc|cc}1 & 0 & -\frac{1}{3} & \frac{2}{3}\\ 0 & 1 & \frac{2}{3} & -\frac{1}{3}\end{array}\right]$$

Thus the matrix of T is:

$$[T] = \left[\left(T^{-1}\right)^{-1}\right] = \left[T^{-1}\right]^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \qquad \blacksquare$$

11. Use the Gauss-Jordan method to find all the solutions, if any, of the system of equations given in question **1**. *[12]*

SOLUTION. As always, we set up the corresponding augmented matrix and set the Gauss-Jordan method upon it:

$$\left[\begin{array}{ccccc|c} 2 & 0 & 4 & 0 & 6 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ 3 & 0 & 5 & 0 & 7 & 0 \end{array}\right] \overset{\frac{1}{2}R_1}{\Longrightarrow} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ 3 & 0 & 5 & 0 & 7 & 0 \end{array}\right]$$

$$\underset{\substack{R_3 - R_1 \\ R_4 - 3R_1}}{\Longrightarrow} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 & -2 & 0 \end{array}\right] \underset{R_3 + R_2}{\Longrightarrow} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 & -2 & 0 \end{array}\right]$$

$$\underset{(-1)R_3}{\Longrightarrow} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & -2 & 0 \end{array}\right] \underset{R_4 + R_3}{\overset{R_1 - 2R_3}{\Longrightarrow}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

In terms of the original variables this corresponds to the equations $u - z = 0$, $v + y = 0$, and $x + 2z = 0$. We set $y = s$ and $z = t$ for parameters s and t; then $u = z = t$, $v = -y = -s$, and $x = -2z = -2t$. It follows that the vector-parametric form of the solutions is

$$\begin{bmatrix} u \\ v \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ -s \\ -2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix},$$

where $s, t \in \mathbb{R}$. $\blacksquare$

[Total = 100]

Part III. Bonus!

0. Write an original little poem about linear algebra or mathematics in general. *[1]*

-5. Give a creative explanation for the lack of a question **5** on this exam. *[1]*

ENJOY THE REST OF THE SUMMER!