Mathematics 1350H – Linear algebra I: Matrix algebra
Trent University, Summer 2014

Assignment #5
Due on Tuesday, 17 June, 2014.

Determinants by way of Gauss-Jordan reduction

Given a square matrix $\mathbf{A}$, we can compute a number called the *determinant* of $\mathbf{A}$, usually denoted by $|\mathbf{A}|$ or $\det(\mathbf{A})$, that gives a lot of information about $\mathbf{A}$. For example, $|\mathbf{A}| \neq 0$ exactly when $\mathbf{A}^{-1}$ exists. One problem with the usual definition of determinants [see §4.2 in the text], which works by reducing the determinant of an $n \times n$ matrix to an alternating sum of determinants of n different $(n-1) \times (n-1)$ sub-matrices, is that computing them this way is a *lot* of work unless $\mathbf{A}$ is a pretty small matrix or has a lot of 0s. (Heck, it's a pain even for 3×3 matrices with the usual definition, as we saw in computing cross-products of vectors in $\mathbb{R}^3$.) In this assignment, we will be looking at a method to compute the determinant of a matrix using the Gauss-Jordan method.

The determinant of an $n \times n$ matrix $\mathbf{A}$ satisfies the following rules:

i. The identity matrix has determinant equal to 1, *i.e.* $|\mathbf{I}_n| = 1$.

ii. If you exchange the ith and jth row of $\mathbf{A}$ to get the matrix $\mathbf{B}$, then $|\mathbf{B}| = -|\mathbf{A}|$.

iii. If you multiply the ith row of $\mathbf{A}$ by a constant c to get the matrix $\mathbf{C}$, then $|\mathbf{C}| = c|\mathbf{A}|$.

iv. If you add a multiple of any row of $\mathbf{A}$ to a different row of $\mathbf{A}$ to get the matrix $\mathbf{D}$, then $|\mathbf{D}| = |\mathbf{A}|$.

v. Taking the transpose of $\mathbf{A}$ doesn't change the determinant. That is, $|\mathbf{A}^T| = |\mathbf{A}|$.

If you really wanted to, by the way, you could actually use this collection of rules as the definition of the determinant of a matrix. It's pretty cumbersome as a definition, but it does provide a much more efficient way to compute the determinant of even a modestly large matrix. It also makes it easier to see why $\mathbf{A}$ is invertible if and only if $|\mathbf{A}| \neq 0$: both are equivalent to the matrix being reducible to $\mathbf{I}_n$ using the Gauss-Jordan method.

1. In both **a** and **b** use the Gauss-Jordan method to put the matrix $\mathbf{A}$ in reduced row-echelon form, and then apply rules *i* – *v* to work out $|\mathbf{A}|$.

a. $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ *[2]*

b. $\mathbf{A} = \begin{bmatrix} 0 & 3 & 6 \\ 2 & 4 & 5 \\ 4 & 7 & 0 \end{bmatrix}$ *[3]*

2. Use rules *i* – *v* to determine $|\mathbf{A}|$ if:

a. $\mathbf{A} = \mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. *[1]*

b. $\mathbf{A}$ has a row of zeros. *[1]*

c. $\mathbf{A}$ has two equal rows. *[1]*

3. Rules *ii* – *iv* are true for the columns of $\mathbf{A}$ as well as the rows. Explain why. *[2]*