Trent University

MATH 1350H Test

3 June, 2013

Time: 50 minutes

Name: Solutions

Now with some corrections!

Student Number: 2.718281

Question	Mark
1	____
2	____
3	____
4	____
Total	____

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Consider the line in $\mathbb{R}^3$ given by the vector equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

a. Find two points on this line. *[1]*

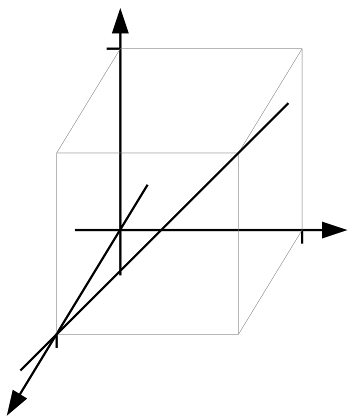
b. Sketch this line. *[2]*

c. Find a vector perpendicular to this line. *[3]*

d. Find the angle between this line and the line given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. *[4]*

SOLUTIONS. **a.** When $t = 0$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and when $t = 1$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so $(1, 0, 0)$ and $(1, 1, 1)$ are two points on the give line. □

b. We plot the points from **a** and connect them:



□

c. $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is a vector in the direction of the line, so we need to find a vector $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ perpendicular to it. Since two vectors are perpendicular exactly when their dot product is zero, this amounts to finding some values for u, v, and w (not all 0) so that

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \end{bmatrix} = 0u + 1v + 1w = v + w = 0\,.$$

It's pretty easy to see that $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, for example, does the job. □

d. (Sanity check: both lines have the same base point, namely $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so asking for the angle between them is meaningful.) The angle θ between the two lines is the angle between their direction vectors, so $\cos(\theta) = \dfrac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\| \left\| \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\|} = \dfrac{0 \cdot 0 + 1 \cdot 1 + 1 \cdot 0}{\sqrt{0^2 + 1^2 + 1^2}\sqrt{0^2 + 1^2 + 0^2}} = \dfrac{1}{\sqrt{2}}$. Since we are looking for an angle between 0 and π radians, and $\cos(\theta)$ is 1-1 on this domain, $\theta = \dfrac{\pi}{4}$. ■

2. Consider the following system of linear equations:

$$\begin{array}{ccccccc} x & + & y & + & z & = & 3 \\ x & - & y & + & z & = & 1 \\ x & + & 3y & + & z & = & k \end{array}$$

a. Find the solution(s), if any, of this system of equations if $k = 2$. *[5]*

b. Find the solution(s), if any, of this system of equations if $k = 5$. *[5]*

SOLUTIONS. To save some overall effort, we'll apply Gauss-Jordan reduction to the given system with the generic k and then sub in the given values of k and see what happens.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 1 & 3 & 1 & k \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & 2 & 0 & k-3 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ -\frac{1}{2}R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & k-3 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \Longrightarrow \\ R_3 - 2R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & k-5 \end{array}\right]$$

Note that the final matrix is in row-reduced echelon form, so we can go no farther.

a. If $k = 2$, the right-hand side entry in the last row is $2 - 5 = -3$, so the line represents the equation $0 = 0x + 0y + 0w = -3$, which is impossible. It follows that there are no solutions to the given system of equations if $k = 2$. □

b. If $k = 5$, the right-hand side entry of the last row is $5 - 5 = 0$, which means that the third equation represented redundant information. What is left only tells us that $x + z = 2$ and $y = 1$. Since z can be anything, we have infinitely many solutions: if t is a parameter and we set $z = t = 0 + t$, then $y = 1 + 0t$ and $x = 2 - z = 2 - t$. The set of all solutions is therefore the line in $\mathbb{R}^3$ given by the vector-parametric equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} . \qquad \blacksquare$$

3. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. Find a linear equation for the plane given by the vector-parametric equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$

b. Find a vector-parametric equation for the plane $2x - y + z = 2$.

c. Find the point(s) of intersection, if any, of the lines in $\mathbb{R}^2$ given by $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x + y = -1$, respectively.

SOLUTIONS. **a.** We need to find a vector normal (*i.e.* perpendicular) to the plane; its entries will serve as the coefficients of the linear equation. The vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ are both parallel to the plane, so their cross-product will be normal to it.

$$\begin{aligned} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = + \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \mathbf{k} \\ &= [1 \cdot (-1) - 0 \cdot 1]\, \mathbf{i} - [1 \cdot (-1) - 2 \cdot 1]\, \mathbf{j} + [1 \cdot 0 - 2 \cdot 1]\, \mathbf{k} \\ &= -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} \end{aligned}$$

Thus the plane has a linear equation of the form $-x + 3y - 2z = d$. To compute d, we plug in the coordinates of the base point of the plane in the given parametric respresentation for x, y, and z: $d = -1(0) + 3(1) - 2(0) = 3$. A linear equation for the given plane is therefore $-x + 3y - 2z = 3$. □

b. We need to find a point on the plane and two vectors parallel to it. We will do so by first finding three points in the plane by setting two of x, y, and z to 0 at a time and solving for the third:

$$\begin{aligned} y = z = 0 &\Longrightarrow 2x - 0 + 0 = 2 \Longrightarrow x = 1 \\ x = z = 0 &\Longrightarrow 2 \cdot 0 - y + 0 = 2 \Longrightarrow y = -2 \\ x = y = 0 &\Longrightarrow 2 \cdot 0 - 0 + z = 2 \Longrightarrow z = 2 \end{aligned}$$

Thus $(1, 0, 0)$, $(0, -2, 0)$, and $(0, 0, 2)$ are three points on the plane. We will use $(1, 0, 0)$ as the base point, and find the vectors we need by computing the vectors that take us from the base point to the other two points:

$$\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

(Note that these two vectors are clearly not multiples of one another. If they were, we'd have to try harder to get a plane.) Thus a vector-parametric equation for the given plane is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}. \quad \square$$

c. We'll plug the expressions for x and y from the line given parametrically into the linear equation for the other line and try to solve for t:

$$-1 = x+y = (0+1t)+(1+1t) = 1+2t \implies 2t = -1-1 = -2 \implies t = -2/2 = -1$$

Plugging $t = -1$ into the parametric reprsentation gives us $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Thus the point $(-1, 0)$ is on both lines. (Note that it does really satisfy $x + y = -1$.) □

4. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. Compute $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{bmatrix}$.

b. If $\mathbf{A}^T\mathbf{B} = \mathbf{I}_{41}$ for some matrices $\mathbf{A}$ and $\mathbf{B}$, what is $\mathbf{B}^T\mathbf{A}$?

c. If $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$, find the vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ such that $\mathbf{Ax} = \mathbf{b}$.

SOLUTIONS. **a.** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot1+2\cdot3 & 1\cdot2+2\cdot2 & 1\cdot3+2\cdot2 \\ 3\cdot1+4\cdot3 & 3\cdot2+4\cdot2 & 3\cdot3+4\cdot2 \\ 5\cdot1+6\cdot3 & 5\cdot2+6\cdot2 & 5\cdot3+6\cdot2 \end{bmatrix} = \begin{bmatrix} 7 & 6 & 7 \\ 15 & 14 & 17 \\ 23 & 22 & 27 \end{bmatrix}$ □

b. Note that $\mathbf{I}_{41}^T = \mathbf{I}_{41}$ and $\left(\mathbf{A}^T\right)^T = \mathbf{A}$. It follows that

$$\mathbf{B}^T\mathbf{A} = \mathbf{B}^T\left(\mathbf{A}^T\right)^T = \left(\mathbf{A}^T\mathbf{B}\right)^T = (\mathbf{I}_{41})^T = \mathbf{I}_{41}^T = \mathbf{I}_{41}\,. \qquad \square$$

c. Since

$$\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \mathbf{b} = \mathbf{Ax} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ x + 2y \end{bmatrix},$$

the problem boils down to solving the two linear equations $2x + y = 6$ and $x + 2y = 6$. Here goes:

$$\left[\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & 2 & 6 \end{array}\right] \underset{\Longrightarrow}{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 2 & 1 & 6 \end{array}\right] \underset{R_2 - 2R_1}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & -3 & -6 \end{array}\right]$$

$$\underset{-\frac{1}{3}R_2}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 1 & 2 \end{array}\right] \underset{\Longrightarrow}{R_1 - 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right]$$

It follows that $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. □

[Total = 40]