Mathematics 1350H – Linear algebra I: Matrix algebra

Solutions to Assignment #4

Determinants the Gauss-Jordan way

Given a square matrix $\mathbf{A}$, we can compute a number called the *determinant* of $\mathbf{A}$, usually denoted by $|\mathbf{A}|$ or $\det(\mathbf{A})$, that gives a lot of information about $\mathbf{A}$. For example, $|\mathbf{A}| \neq 0$ exactly when $\mathbf{A}^{-1}$ exists. One problem with the usual definition of determinants – which works by reducing the determinant of an $n \times n$ matrix to a weighted sum of n determinants of $(n-1) \times (n-1)$ matrices - is that computing them this way is a *lot* of work unless $\mathbf{A}$ is a pretty small matrix. (Heck, it's a pain even for 3×3 matrices with the usual definition ...) Here are some facts which let you compute the determinant of a matrix using the Gauss-Jordan method:

The determinant of an $n \times n$ matrix $\mathbf{A}$ satisfies the following rules:

i. The identity matrix has determinant equal to 1, *i.e.* $|\mathbf{I}_n| = 1$.

ii. If you exchange the ith and jth row of $\mathbf{A}$ to get the matrix $\mathbf{B}$, then $|\mathbf{B}| = -|\mathbf{A}|$.

iii. If you multiply the ith row of $\mathbf{A}$ by a constant c to get the matrix $\mathbf{C}$, then $|\mathbf{C}| = c|\mathbf{A}|$.

iv. If you add a multiple of any row of $\mathbf{A}$ to a different row of $\mathbf{A}$ to get the matrix $\mathbf{D}$, then $|\mathbf{D}| = |\mathbf{A}|$. (In general, if you add any row vector $\mathbf{r}$ to the ith row of $\mathbf{A}$ to get the matrix $\mathbf{D}$, then $|\mathbf{D}| = |\mathbf{A}| + |\mathbf{A}_{i,\mathbf{r}}|$, where $\mathbf{A}_{i,\mathbf{r}}$ is the matrix $\mathbf{A}$ with its ith row replaced by $\mathbf{r}$.)

v. Taking the transpose of $\mathbf{A}$ doesn't change the determinant. That is, $|\mathbf{A}^T| = |\mathbf{A}|$.

If you really wanted to, by the way, you could actually use this collection of rules as the definition of the determinant of a matrix. It's pretty cumbersome as a definition, but it does provide a much more efficient way to compute the determinant of even a modestly large matrix.

1. Use rules *i* – *v* ~~, as well as **1** and **2**,~~ to compute $|\mathbf{A}|$ if:

a. $\mathbf{A}$ has a column or a row of zeros. *[1.5]*

b. $\mathbf{A}$ has two equal columns or two equal rows. *[1.5]*

c. $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$. *[2]*

Solutions. **a.** Suppose $\mathbf{A}$ is an $n \times n$ matrix whose ith row, call it $\mathbf{r}_i$, is all zeros. Note that in this case $\mathbf{r}_i = 0\mathbf{r}_i$, so, by rule *iii*, $|\mathbf{A}| = 0|\mathbf{A}| = 0$.

If $\mathbf{A}$ has a column of zeros instead, then $\mathbf{A}^T$ must have a row of zeros, so $|\mathbf{A}| = |\mathbf{A}^T| = 0$, by the above and rule *v*. □

b. Suppose $\mathbf{A}$ is a matrix whose ith and jth rows are the same (with $i \neq j$, of course). Then $\mathbf{A} \underset{R_i \leftrightarrow R_j}{\Longrightarrow} \mathbf{A}$, so, by rule *ii*, $|\mathbf{A}| = -|\mathbf{A}|$. The only number which is equal to its own negative is 0, so it must be the case that $|\mathbf{A}| = 0$. □

c. We'll put $\mathbf{A}$ in row-reduced echelon form and then figure out $|\mathbf{A}|$ by applying the rules.

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \underset{\Longrightarrow}{\tfrac{1}{3}R_1} \begin{bmatrix} 1 & \frac{4}{3} \\ 5 & 6 \end{bmatrix} \underset{R_2 - 5R_1}{\Longrightarrow} \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{2}{3} \end{bmatrix} \underset{-\frac{3}{2}R_2}{\Longrightarrow} \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix} \underset{\Longrightarrow}{R_1 - \tfrac{4}{3}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The final, row-reduced, matrix is just $\mathbf{I}_2$, which has determinant 1 by rule i. It was obtained from $\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}$ by subtracting a multiple of one row from another, which does not change the determinant by rule iv, so $\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}$ also has determinant 1. This matrix, in turn, was obtained from $\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{2}{3} \end{bmatrix}$ by multiplying a row by $-\frac{3}{2}$, which changes the determinant by a factor of $-\frac{3}{2}$ by rule iii, *i.e.* $1 = \left|\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}\right| = -\frac{3}{2}\left|\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{2}{3} \end{bmatrix}\right|$. It follows that $\left|\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{2}{3} \end{bmatrix}\right| = 1 \div \left(-\frac{3}{2}\right) = -\frac{2}{3}$. Since $\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{2}{3} \end{bmatrix}$ was obtained from $\begin{bmatrix} 1 & \frac{4}{3} \\ 5 & 6 \end{bmatrix}$ by subtracting a multiple of one row from another, rule iv tells us that $\left|\begin{bmatrix} 1 & \frac{4}{3} \\ 5 & 6 \end{bmatrix}\right| = -\frac{2}{3}$ too. $\begin{bmatrix} 1 & \frac{4}{3} \\ 5 & 6 \end{bmatrix}$ was obtained from our original matrix $\mathbf{A}$ by multiplying a row by $\frac{1}{3}$, so $-\frac{2}{3} = \left|\begin{bmatrix} 1 & \frac{4}{3} \\ 5 & 6 \end{bmatrix}\right| = \frac{1}{3}|\mathbf{A}|$ by rule iii. Thus $|\mathbf{A}| = \left(-\frac{2}{3}\right) \div \frac{1}{3} = -2$. [Whew!] □

2. Rules ii – iv are true for the columns of $\mathbf{A}$ as well as the rows. Why? *[2]*

SOLUTION. Rule v is the reason[1]. Applying the operations mentioned in rules ii – iv to the columns of $\mathbf{A}$ corresponds to applying them to the rows of $\mathbf{A}^T$. Rule v tells us that $|\mathbf{B}| = |\mathbf{B}^T|$ for any matrix $\mathbf{B}$, so the effect on $|\mathbf{A}|$ of column operations on $\mathbf{A}$ is exactly the same as the effect on $|\mathbf{A}^T|$ of the corresponding row operations on $\mathbf{A}^T$. Hence rules ii – iv work for columns as well as rows. ■

3. Use the Gauss-Jordan method to put the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 0 \end{bmatrix}$ in reduced row-echelon form. Apply what you have learned above to use this computation to determine $|\mathbf{A}|$. *[3]*

SOLUTION. We'll use the same method as for **1c** above, though we won't be quite so painstaking in tracing how the determinant changes during the computation. First, the full Gauss-Jordan:

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 0 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\Longrightarrow} \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 7 & 0 \end{bmatrix} \overset{\frac{1}{3}R_1}{\Longrightarrow} \begin{bmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 0 & 1 & 2 \\ 6 & 7 & 0 \end{bmatrix}$$

$$\underset{R_3 - 6R_1}{\Longrightarrow} \begin{bmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 0 & 1 & 2 \\ 0 & -1 & -10 \end{bmatrix} \underset{R_3 + R_2}{\overset{R_1 - \frac{4}{3}R_2}{\Longrightarrow}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -8 \end{bmatrix} \underset{-\frac{1}{8}R_3}{\Longrightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

[1] But only when rows are not in season! [With apologies to Tom Lehrer.]

$$\begin{array}{c} R_1 + R_3 \\ R_2 - 2R_3 \\ \Longrightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row-reduced matrix has determinant 1 by rule i. The only row operations which changed the determinant were the swap of two rows and the multiplication of rows by $\frac{1}{3}$ and $-\frac{1}{8}$, repectively. It follows that $\left(-\frac{1}{8}\right)\left(\frac{1}{3}\right)(-1)|\mathbf{A}| = 1$, so it must be the case that $|\mathbf{A}| = 1 \div \left(-\frac{1}{8}\right)\left(\frac{1}{3}\right)(-1) = 24$. ■

Bonus. Assuming the general part of rule *iv* (the part in parentheses) is true, show that the particular part of rule *iv* (the part not in parentheses) must be true. You may use the other rules as well. *[2]*

SOLUTION. Suppose we obtain $\mathbf{E}$ by adding c times row i of $\mathbf{A}$ to row j of $\mathbf{A}$. (That is, $\mathbf{A} \underset{R_j + cR_i}{\Longrightarrow} \mathbf{E}$.) Suppose $\mathbf{C}$ is the matrix $\mathbf{A}$ with row j replaced by c times row i, and $\mathbf{B}$ is the matrix $\mathbf{A}$ with row j replaced by row i. Then $|\mathbf{E}| = |\mathbf{A}| + |\mathbf{C}|$ (by rule *iv*) $= |\mathbf{A}| + c|\mathbf{B}|$ (by rule *iii*). Since $|\mathbf{B}| = 0$ by **1b**, it follows that $|\mathbf{E}| = |\mathbf{A}|$. ■