Mathematics 1350H – Linear algebra I: Matrix algebra
TRENT UNIVERSITY, Summer 2013
SOLUTIONS TO ASSIGNMENT #3

Complex relationships

Let $\mathbf{I}_n$ denote the $n \times n$ identity matrix.

1. Find a 2×2 matrix $\mathbf{T}$ such that $\mathbf{T}^2 = -\mathbf{I}_2$. *[4]*

SOLUTION. There are a lot of ways to get the job done. One of the simplest matrices that works is $\mathbf{T} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Note that if $\mathbf{T}^2 = -\mathbf{I}_2$, then $(-\mathbf{T})^2 = -\mathbf{I}_2$ as well. (Why?) ■

2. Find 4×4 matrices U, V, and W such that $\mathbf{U}^2 = \mathbf{V}^2 = \mathbf{W}^2 = -\mathbf{I}_4$, $\mathbf{UV} = \mathbf{W}$, $\mathbf{VW} = \mathbf{U}$, $\mathbf{WU} = \mathbf{V}$, $\mathbf{VU} = -\mathbf{W}$, $\mathbf{WV} = -\mathbf{U}$, and $\mathbf{UW} = -\mathbf{V}$. *[6]*

HINT: You can use the matrix $\mathbf{T}$ from your solution to problem **1** as a submatrix of at least one of the matrices you need to build for problem **2**.

SOLUTION. Following the hint, or applying the idea behind the solution to **1** on a larger scale, let

$$\mathbf{U} = \begin{bmatrix} \mathbf{0} & \mathbf{T} \\ \mathbf{T} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} -\mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then we must have

$$\mathbf{W} = \mathbf{UV} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and we'll leave it to the interested reader to check that all the matrix equations that must be satisfied actually are. ■