Mathematics 1350H – Linear algebra I: Matrix algebra

TRENT UNIVERSITY, Summer 2013

SOLUTIONS TO ASSIGNMENT #2

Planes on a gem?

Consider the planes in $\mathbb{R}^3$ given by the equations $x+y-z=2$, $x-y-z=2$, $-x+y-z=2$, $-x-y-z=2$, $x+y+z=2$, $x-y+z=2$, $-x+y+z=2$, $-x-y+z=2$, $z=-1$, and $z=1$.

1. Find all the points where three or more of these planes intersect. *[8]*

SOLUTION. Zeroth, the brute force approach to this problem would be to simply take every possible set of three out of the ten given planes and try to solve for the point where they meet, if there is one. Since there are $\binom{10}{3}=120$ possible ways to select three planes out of ten, the brute force approach is not recommended. Fortunately, a little examination and experimentation can produce significant shortcuts.

First, observe that the ten planes come in five pairs of parallel planes:

- $x+y-z=2$ and $-x-y+z=2$
- $x-y-z=2$ and $-x+y+z=2$
- $-x+y-z=2$ and $x-y+z=2$
- $-x-y-z=2$ and $x+y+z=2$
- $z=-1$ and $z=1$

No group of three that includes a pair of parallel planes can have a common point of intersection. (This cuts us down to dealing with only 80 possible groups of three ...)

Second, note that the equations of the first four pairs of planes have a lot in common. The left-hand sides of the given equations run through all the possibilities giving x, y, and z coefficients of ± 1, and the right-hand sides are all 2. One immediate consequence is that each of these planes intercepts each axis at 2 or -2; for example, $x+y-z=2$ has intercepts $x=2$, $y=2$, and $z=-2$. This hands us six points, at each of which four of the eight planes meet:

- $x+y-z=2$, $x-y-z=2$, $x-y+z=2$, and $x+y+z=2$ meet at $(2,0,0)$.
- $x+y-z=2$, $-x+y+z=2$, $x-y+z=2$, and $x+y+z=2$ meet at $(0,2,0)$.
- $-x-y+z=2$, $-x+y+z=2$, $x-y+z=2$, and $x+y+z=2$ meet at $(0,0,2)$.
- $-x-y+z=2$, $-x+y+z=2$, $-x+y-z=2$, and $-x-y-z=2$ meet at $(-2,0,0)$.
- $-x-y+z=2$, $x-y-z=2$, $x-y+z=2$, and $-x-y-z=2$ meet at $(0,-2,0)$.
- $x+y-z=2$, $x-y-z=2$, $-x+y-z=2$, and $-x-y-z=2$ meet at $(0,0,-2)$.

It is not hard to check (and should be obvious from the picture answering problem **2** below) that each of these six intercepts is the only common point of any three of the four planes that meet there. (Note that the observations above neatly save us from having to compute the solutions to 36 possible groups of three equations ...)

Third, we need to consider how the last two planes, $z=1$ and $z=-1$, intersect with the other eight. As a prototype, let's see how $z=1$ meets the four planes $x+y-z=2$,

$x-y-z=2$, $x-y+z=2$, and $x+y+z=2$ that have a common point at $(2,0,0)$ [the – easy! – calculations are left out]:

- $z=1$, $x+y-z=2$, and $x-y-z=2$ meet at $(3,0,1)$.
- $z=1$, $x+y-z=2$, and $x-y+z=2$ meet at $(2,1,1)$.
- $z=1$, $x+y-z=2$, and $x+y+z=2$ do not have a common point.
- $z=1$, $x-y-z=2$, and $x-y+z=2$ do not have a common point.
- $z=1$, $x-y-z=2$, and $x+y+z=2$ meet at $(2,-1,1)$.
- $z=1$, $x-y+z=2$, and $x+y+z=2$ meet at $(1,0,1)$.

Similarly, $z=-1$ meets the four planes with a common point at $(2,0,0)$ as follows:

- $z=-1$, $x+y-z=2$, and $x-y-z=2$ meet at $(1,0,-1)$.
- $z=-1$, $x+y-z=2$, and $x-y+z=2$ meet at $(2,-1,-1)$.
- $z=-1$, $x+y-z=2$, and $x+y+z=2$ do not have a common point.
- $z=-1$, $x-y-z=2$, and $x-y+z=2$ do not have a common point.
- $z=-1$, $x-y-z=2$, and $x+y+z=2$ meet at $(2,1,-1)$.
- $z=-1$, $x-y+z=2$, and $x+y+z=2$ meet at $(3,0,-1)$.

Continuing on, $z=\pm 1$ meet the four planes with a common point at $(-2,0,0)$ as follows:

- $z=1$ and $z=-1$ meet the pair $-x+y-z=2$ and $-x-y-z=2$ at $(-3,0,1)$ and $(-1,0,-1)$, respectively.
- $z=1$ and $z=-1$ meet the pair $-x+y-z=2$ and $-x-y+z=2$ at $(-2,1,1)$ and $(-2,-1,-1)$, respectively.
- Neither of $z=1$ and $z=-1$ has a point in common with the pair $-x+y-z=2$ and $-x+y+z=2$.
- Neither of $z=1$ and $z=-1$ has a point in common with the pair $-x-y-z=2$ and $-x-y+z=2$.
- $z=1$ and $z=-1$ meet the pair $-x-y-z=2$ and $-x+y+z=2$ meet at $(-2,-1,1)$ and $(-2,1,-1)$, respectively.
- $z=1$ and $z=-1$ meet the pair $-x-y+z=2$, and $-x+y+z=2$ meet at $(-1,0,1)$ and $(-3,0,-1)$, respectively.

Continuing further, we consider how $z=\pm 1$ meet the eight planes with a common point at $(0,\pm 2,0)$. These cases are just like the ones above, except that x and y trade roles. This gives us the sixteen points – you can figure out which of the planes intersect in each for yourselves – $(0,\pm 3,\pm 1)$, $(0,\pm 1,\pm 1)$, and $(\pm 1,\pm 2,\pm 1)$.

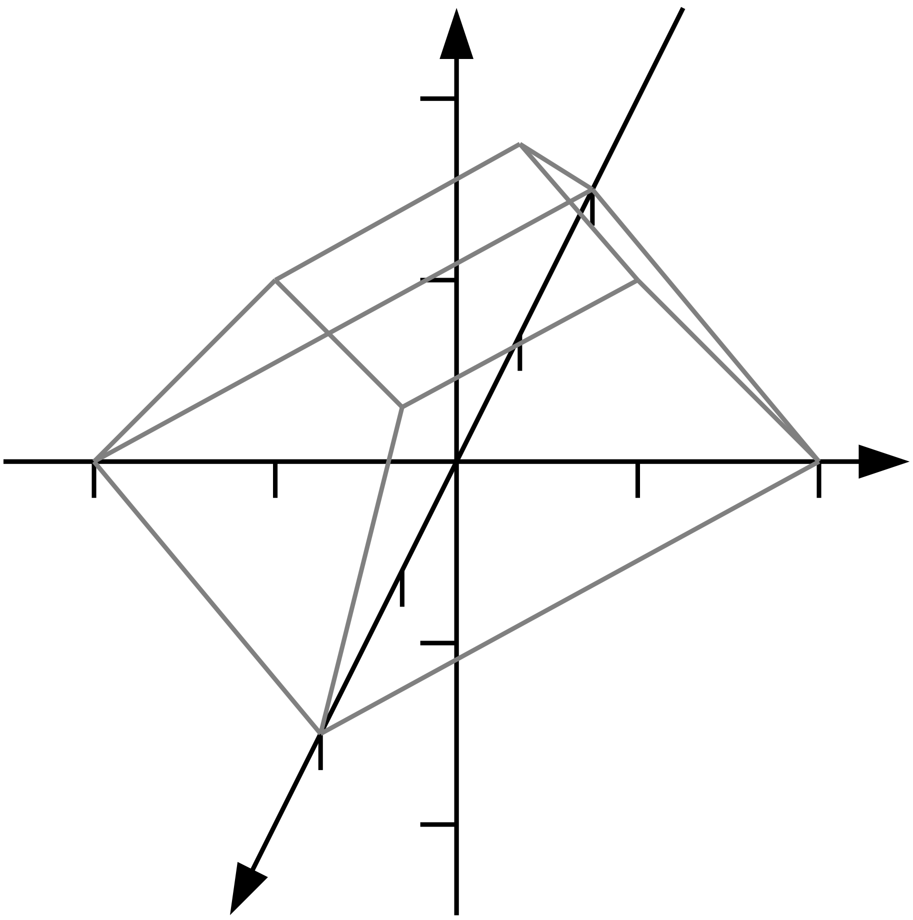
Between the various cases considered above, we have also take care of how $z=\pm 1$ meet the eight planes with a common point at $(0,0,\pm 2)$. Thus, the points where three (or more) of the given planes meet are:

- $(\pm 2,0,0)$, $(0,\pm 2,0)$, and $(0,0,\pm 2)$
- $(\pm 3,0,\pm 1)$, $(\pm 1,0,\pm 1)$, and $(\pm 2,\pm 1,\pm 1)$
- $(0,\pm 3,\pm 1)$, $(0,\pm 1,\pm 1)$, and $(\pm 1,\pm 2,\pm 1)$

Whew! ■

2. Sketch the convex solid containing the origin each of whose faces is a piece of one these planes. *[2]*

Solution. Here is a crude sketch of the top half only, to reduce clutter. Note that the bottom half is symmetric; more precisely, it is a reflection of the top half in the plane $z = 0$.



The solid as a whole is two square-based pyramids, with their tops cut off, stuck together base-to-base. ■