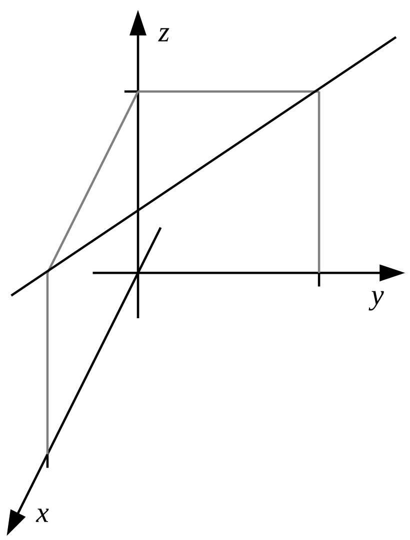
Mathematics 1350H – Linear algebra I: Matrix algebra
TRENT UNIVERSITY, Summer 2013

Solutions to the Quizzes

Quiz #1. Wednesday, 15 May, 2013. *[10 minutes]*

1. Draw a sketch of the points $(1, 0, 1)$ and $(0, 1, 1)$, and the line joining them. (A crude sketch will suffice. :-) *[1.5]*
2. Find a vector parallel to the line. *[1.5]*
3. Determine whether the vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is perpendicular to the line or not. *[2]*

SOLUTION TO 1. Here's a crude sketch:



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SOLUTION TO 2. The vector that takes us from $(1, 0, 1)$ to $(0, 1, 1)$, namely

$$\begin{bmatrix} 0-1 \\ 1-0 \\ 1-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

must be parallel to the line joining the two points. ■

SOLUTION TO 3. Two vectors are perpendicular exactly when their dot products are 0. Since

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 1 \cdot (-1) + 1 \cdot 1 + 0 \cdot 0 = -1 + 1 + 0 = 0\,,$$

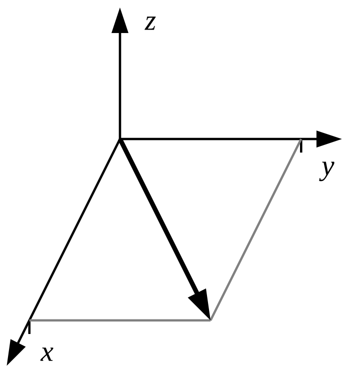
the vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is perpendicular to the vector $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Since the latter vector is parallel to the line, this means that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is indeed perpendicular to the line. ■

Quiz #2. Wednesday, 22 May, 2013. *[10 minutes]*

Let $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

1. Find a vector $\mathbf{u}$ such that the angle between $\mathbf{i}$ and $\mathbf{u}$ is $\frac{\pi}{4}$ radians (*i.e.* 45°). *[3]*
2. Verify that the angle between $\mathbf{i}$ and $\mathbf{u}$ really is $\frac{\pi}{4}$ radians. *[2]*

SOLUTION TO 1. $\mathbf{i}$ points along the positive x-axis, which is perpendicular – that is, is at an angle of $\frac{\pi}{2}$ radians – to the y-axis. The line given by the equation $y = x$ in the xy-plane (*i.e.* the plane $z = 0$) is at half this angle to both axes. Thus any vector parallel to this line, say $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, will have an angle between itself and $\mathbf{i}$ of $\frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$ radians.



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NOTE. There are other ways to solve question 1, of course, but we can draw a picture for the geometrical approach above ...

SOLUTION TO 2. Let θ be the acute angle between the vector $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and the vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ obtained above. (We'll use radian measure for θ throughout in this solution.) Then

$$\cos(\theta) = \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\| \left\| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\|} = \frac{1 \cdot 1 + 0 \cdot 1 + 0 \cdot 0}{\sqrt{1^2 + 0^2 + 0^2}\sqrt{1^2 + 1^1 + 0^2}} = \frac{1}{\sqrt{1}\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Since $0 \leq \theta \leq \pi$ and $\cos(\theta) = \frac{1}{\sqrt{2}}$, it follows that $\theta = \frac{\pi}{4}$. ■

Quiz #3. Monday, 27 May, 2013. *[15 minutes]*

1. Find all the solutions, if any, to the following system of linear equations:

$$\begin{array}{rcrcrcl} 2x & + & 3y & + & z & = & 6 \\ -x & + & y & + & z & = & 1 \\ 3x & - & y & - & z & = & 1 \end{array} \qquad [5]$$

SOLUTION. We'll go whole hog and apply the Gauss-Jordan method, using matrix notation:

$$\left[\begin{array}{rrr|r} 2 & 3 & 1 & 6 \\ -1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ -R_2 \\ \end{array} \left[\begin{array}{rrr|r} 2 & 3 & 1 & 6 \\ 1 & -1 & -1 & -1 \\ 3 & -1 & -1 & 1 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{rrr|r} 1 & -1 & -1 & -1 \\ 2 & 3 & 1 & 6 \\ 3 & -1 & -1 & 1 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{rrr|r} 1 & -1 & -1 & -1 \\ 0 & 5 & 3 & 8 \\ 0 & 2 & 2 & 4 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ \frac{1}{2}R_3 \end{array} \left[\begin{array}{rrr|r} 1 & -1 & -1 & -1 \\ 0 & 5 & 3 & 8 \\ 0 & 1 & 1 & 2 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 \leftrightarrow R_3 \\ \end{array} \left[\begin{array}{rrr|r} 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 5 & 3 & 8 \end{array}\right]$$

$$\begin{array}{c} R_1 + R_2 \\ \Longrightarrow \\ R_3 - 5R_2 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -2 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ -\frac{1}{2}R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - R_3 \\ \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

It follows that there is exactly one solution, namely $x = y = z = 1$, to the given system of linear equations. ■

Quiz #4. Wednesday, 29 May, 2013. *[15 minutes]*

1. Use Gauss-Jordan reduction to find all the solutions, if any, to the following system of linear equations:

$$\begin{array}{ccccccccc} w & + & x & + & y & & & = & 1 \\ w & & & + & 2y & & & = & 1 \\ w & & & + & y & + & z & = & 1 \\ w & + & x & & & + & z & = & 1 \end{array} \qquad \textit{[5]}$$

Solution. Here goes!

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \begin{array}{c} \Longrightarrow \\ -R_2 \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ \Longrightarrow \\ R_3 + R_2 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ -R_3 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \begin{array}{c} R_1 - 2R_3 \\ R_2 + R_3 \\ \Longrightarrow \\ R_4 + R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

It follows that system has infinitely many solutions. In particular, if we set $z = t$ for some parameter t, then $y = z = t$, $x = y = t$, and $w = 1 - 2z = 1 - 2t$. In vector form, this is the line in $\mathbb{R}^4$ given by

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix} . \qquad \blacksquare$$

Quiz #5. Wednesday, 5 June, 2013. *[10 minutes]*

1. Find the inverse, if any, of the following matrix:

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 3 \\ 2 & 1 & -1 \end{bmatrix} \qquad \textit{[5]}$$

SOLUTION. As usual, we set up the super-augmented matrix and apply the Gauss-Jordan algorithm:

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ (-1)R_2 \end{array} \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & -2 & -3 & 0 & -1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1 \leftrightarrow R_2 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & -1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & -1 & 0 \\ 0 & 7 & 7 & 1 & 2 & 0 \\ 0 & 5 & 5 & 0 & 2 & 1 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ \frac{1}{7}R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & -1 & 0 \\ 0 & 1 & 1 & \frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 5 & 5 & 0 & 2 & 1 \end{array}\right] \begin{array}{c} R_1 + 2R_2 \\ \Longrightarrow \\ R_3 - 5R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{2}{7} & -\frac{3}{7} & 0 \\ 0 & 1 & 1 & \frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & -\frac{5}{7} & \frac{4}{7} & 1 \end{array}\right]$$

Since we have reached a state where there is a row of 0s on the left-hand side of the super-augmented matrix, we cannot get to a copy of the identity matrix on the left-hand side. It follows, in particular, that the original matrix has no inverse. ■

Quiz #6. Monday, 10 June, 2013. *[10 minutes]*

1. Let $M = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$. Find a minimal spanning subset of M, *i.e.* a set $P \subseteq M$ that is as small as possible and such that $\mathrm{Span}(P) = \mathrm{Span}(M)$. *[5]*

SOLUTION. We'll assemble the vectors in M into the columns of a matrix, reduce the matrix, and then pick out the vectors in M whose columns in the reduced matrix have a leading 1 for some row.

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ 1 & 5 & 3 & 2 \\ 3 & 5 & 4 & 1 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\Longrightarrow} \begin{bmatrix} 1 & 5 & 3 & 2 \\ 2 & 2 & 2 & 0 \\ 3 & 5 & 4 & 1 \end{bmatrix}$$

$$\begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \begin{bmatrix} 1 & 5 & 3 & 2 \\ 0 & -8 & -4 & -4 \\ 0 & -10 & -5 & -5 \end{bmatrix} \begin{array}{c} \Longrightarrow \\ -\frac{1}{8}R_2 \\ \end{array} \begin{bmatrix} 1 & 5 & 3 & 2 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -10 & -5 & -5 \end{bmatrix}$$

$$\begin{array}{c} R_1 - 5R_2 \\ \Longrightarrow \\ R_3 + 10R_2 \end{array} \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that $P = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix} \right\}$ is a minimal spanning subset of M. ∎

Quiz #7. Wednesday, 12 June, 2013. *[12 minutes]*

Let $\mathbf{A} = \begin{bmatrix} 2 & -2 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 3 & 1 & 0 & 5 \end{bmatrix}$.

1. Use the Gauss-Jordan method to put $\mathbf{A}$ in row-reduced echelon form. *[2]*

2. Find a basis for *two* (2) of the following three subspaces:

i. col($\mathbf{A}$) *ii.* row($\mathbf{A}$) *iii.* null($\mathbf{A}$) *[3 = 2 × 1.5 each]*

Solution to 1. Here goes:

$$\begin{bmatrix} 2 & -2 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 3 & 1 & 0 & 5 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\Longrightarrow} \begin{bmatrix} -1 & 1 & 0 & 1 \\ 2 & -2 & 1 & 0 \\ 3 & 1 & 0 & 5 \end{bmatrix} \overset{(-1)R_1}{\Longrightarrow} \begin{bmatrix} 1 & -1 & 0 & -1 \\ 2 & -2 & 1 & 0 \\ 3 & 1 & 0 & 5 \end{bmatrix}$$

$$\begin{matrix} \Longrightarrow \\ R_2 - 2R_2 \\ R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 4 & 0 & 8 \end{bmatrix} \overset{\Longrightarrow}{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 4 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix} \overset{\Longrightarrow}{\frac{1}{4}R_2} \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\overset{R_1 + R_2}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

That's all folks! ■

Solution to 2. *i.* The columns of the original matrix corresponding to the columns in which the reduced matrix has a leading 1 for some row form a basis for the column space of the matrix. Thus $\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for col($\mathbf{A}$). □

ii. The non-zero rows of the reduced matrix form a basis for the row space of the original matrix. Thus, writing the rows as column vectors, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for row($\mathbf{A}$). □

iii. This one we actually have to work for a little … If $\mathbf{B}$ is the reduced matrix, then null($\mathbf{A}$) $= \{ \mathbf{x} \mid \mathbf{Ax} = \mathbf{0} \} = \{ \mathbf{x} \mid \mathbf{Bx} = \mathbf{0} \}$. (Note that reducing a system of equations does not change what the solutions are.) If $\mathbf{x} = [w \quad x \quad y \quad z]^T$, then $\mathbf{Bx} = \mathbf{0}$ boils down to the equations $w + z = 0$, $x + 2z = 0$, and $y + 2z = 0$. Setting $z = t$ for a parameter t and solving for the other variables in terms of t gives

$$\mathbf{x} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -2 \\ -2 \\ 1 \end{bmatrix}.$$

It follows that $\left\{ \begin{bmatrix} -1 \\ -2 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for null($\mathbf{A}$). ■

Quiz #8. Monday, 17 June, 2013. *[15 minutes]*

$$\text{Let } W = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

1. Find an orthogonal basis for W. *[5]*

SOLUTION. First, we find a basis for W. We assemble the vectors whose span W is into the columns of a matrix and then reduce this matrix:

$$\begin{bmatrix} 1 & 1 & 3 & -1 \\ -1 & 1 & -1 & -3 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix} \begin{matrix} \Longrightarrow \\ R_2 + R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 2 & 2 & -4 \\ 0 & -1 & -1 & 2 \\ 0 & -2 & -2 & 4 \end{bmatrix}$$

$$\begin{matrix} \frac{1}{2}R_2 \\ \Longrightarrow \end{matrix} \begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & -1 & 2 \\ 0 & -2 & -2 & 4 \end{bmatrix} \begin{matrix} R_1 - R_1 \\ \Longrightarrow \\ R_3 + R_2 \\ R_4 + 2R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The vectors corresponding to the columns in which leading 1s occur in the reduced matrix will do; that is, $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a basis for W.

Second, we make our basis orthogonal. We can take the first basis vector unchanged, but we have to make the second basis vector orthogonal to it by subtracting the component of it that is parallel to the first:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{-1}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ \frac{3}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{bmatrix}$$

Thus $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{5}{4} \\ \frac{3}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{bmatrix} \right\}$ is an orthogonal matrix for W. ■

Quiz #9. Wednesday, 19 June, 2013. *[10 minutes]*

A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix} \text{ and } T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}.$$

1. Find $T\left(\begin{bmatrix}3\\4\end{bmatrix}\right)$. *[5]*

Solution. We will first write $\begin{bmatrix}3\\4\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$, and then use the linearity of T and its known values to compute $T\left(\begin{bmatrix}3\\4\end{bmatrix}\right)$.

First, $\begin{bmatrix}3\\4\end{bmatrix} = a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}2\\1\end{bmatrix}$ if and only if $a + 2b = 3$ and $2a + b = 4$. We solve this system of linear equations in the usual way:

$$\begin{bmatrix}1 & 2 & 3\\2 & 1 & 4\end{bmatrix} \underset{R_2 - 2R_1}{\Longrightarrow} \begin{bmatrix}1 & 2 & 3\\0 & -3 & -2\end{bmatrix} \underset{-\frac{1}{3}R_2}{\Longrightarrow} \begin{bmatrix}1 & 2 & 3\\0 & 1 & \frac{2}{3}\end{bmatrix} \underset{\Longrightarrow}{R_1 - 2R_2} \begin{bmatrix}1 & 0 & \frac{5}{3}\\0 & 1 & \frac{2}{3}\end{bmatrix}$$

Thus $\begin{bmatrix}3\\4\end{bmatrix} = \frac{5}{3}\begin{bmatrix}1\\2\end{bmatrix} + \frac{2}{3}\begin{bmatrix}2\\1\end{bmatrix}$. It now follows, by the linearity of T, that

$$\begin{aligned} T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) = T\left(\frac{5}{3}\begin{bmatrix}1\\2\end{bmatrix} + \frac{2}{3}\begin{bmatrix}2\\1\end{bmatrix}\right) &= \frac{5}{3}T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) + \frac{2}{3}T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) \\ &= \frac{5}{3}\begin{bmatrix}0\\1\end{bmatrix} + \frac{2}{3}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\\frac{5}{3}\end{bmatrix}. \quad \blacksquare \end{aligned}$$