Mathematics 1350H – Linear algebra I: Matrix algebra
TRENT UNIVERSITY, Summer 2013

FINAL EXAMINATION
Friday, 19 June, 2013

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Do parts **Y** and **Z**. Show all your work. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; $\leq 10^{10^{10}}$ neurons.

Part Y. Do all of **1**–**5**. *[Subtotal = 64/100]*

1. Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 & 2 & 2 & 3 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 4 & 4 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$.

- **a.** Without any calculation, does the equation $\mathbf{Ax} = \mathbf{0}$ have no solutions, just one solution, or many solutions? Explain why. *[2]*
- **b.** Use the Gauss-Jordan method to put $\mathbf{A}$ in reduced row-echelon form. *[10]*
- **c.** What are the rank and nullity of $\mathbf{A}$? *[1]*
- **d.** Without any further calculation, give a basis for $\mathrm{col}(\mathbf{A})$. *[3]*
- **e.** Find a basis for $\mathrm{null}(\mathbf{A})$. *[4]*

2. Consider the line in $\mathbb{R}^3$ passing through the points $(2,2,0)$ and $(2,0,2)$, and also the line passing through the points $(0,1,1)$ and $(1,1,1)$.

- **a.** Sketch these points and lines. *[2]*
- **b.** Find a parametric description of each of these lines. *[4]*
- **c.** Find the point at which the two lines meet and the (smallest) angle between them at that point. *[4]*

3. Let $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

- **a.** Find $\mathbf{B}^{-1}$, if it exists. *[10]*
- **b.** Use your work in part **a** to compute $|\mathbf{B}|$. *[5]*

4. Find an equation of the form $ax + by + cz = d$ for the plane containing both the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and the point $(1,1,1)$. *[9]*

5. Let $\mathbf{D} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

- **a.** Find all the eigenvalues of $\mathbf{D}$. *[5]*
- **b.** Find all the eigenvectors of $\mathbf{D}$. *[5]*

[Parts **Z** *and* $\heartsuit$ *are on page 2.]*

Part Z. Do any *three* of **6–11**. *[Subtotal = 36/100]*

6. Use the properties of the vector operations and the dot product to verify that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}\left(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2\right)$. *[12]*

7. Determine whether $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, |x| = |y| \right\}$ a subspace of $\mathbb{R}^2$ or not. If it is a subspace, determine its dimension. *[12]*

8. Consider the planes in $\mathbb{R}^3$ given by the equations $2x + 2y + z = 6$ and $x - y = 0$, respectively.

- **a.** Give a parametric description of the line of intersection of these two planes. *[8]*
- **b.** Find the points, if any. in which the line given by $x = t$, $y = 3 - t$, and $z = 1$ intersects each of the two planes. *[4]*

9. Find a 2×2 matrix $\mathbf{X}$ such that $\mathbf{X}^2 - 2\mathbf{X} + \mathbf{I}_2 = \mathbf{O}_2$, where $\mathbf{O}_2$ is the 2×2 zero matrix. Is there another such $\mathbf{X}$? Explain why or why not. *[12]*

10. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that

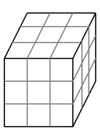
$$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ T\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \text{and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

a. Find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. *[4]* **b.** Compute $T\left(\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}\right)$. *[8]*

11. Find an orthogonal basis for $U = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. *[12]*

[Total = 100]

Part ♡. Bonus!



$\ddot\smile$. A dangerously sharp tool is used to cut a cube with a side length of $3\,cm$ into 27 smaller cubes with a side length of $1\,cm$. This can be done easily with six cuts. Can it be done with fewer? (You may rearrange the pieces between cuts.) If so, explain how; if not, explain why not. *[1]*

$\ddot\smile$. Write an original little poem about linear algebra or mathematics in general. *[2]*

Enjoy the rest of the summer!