Mathematics 1350H – Linear algebra I: Matrix algebra

Trent University, Summer 2013

Solutions to the Final Examination

Friday, 19 June, 2013

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Do parts **Y** and **Z**. Show all your work. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; $\leq 10^{10^{10}}$ neurons.

Part Y. Do all of **1**–**5**. *[Subtotal = 64/100]*

1. Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 & 2 & 2 & 3 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 4 & 4 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$.

- **a.** Without any calculation, does the equation $\mathbf{Ax} = \mathbf{0}$ have no solutions, just one solution, or many solutions? Explain why. *[2]*
- **b.** Use the Gauss-Jordan method to put $\mathbf{A}$ in reduced row-echelon form. *[10]*
- **c.** What are the rank and nullity of $\mathbf{A}$? *[1]*
- **d.** Without any further calculation, give a basis for col($\mathbf{A}$). *[3]*
- **e.** Find a basis for null($\mathbf{A}$). *[4]*

Solutions. **a.** Since the third and fifth rows are multiples of one another, the rank of the 5×5 matrix $\mathbf{A}$ can be no more than $5 - 1 = 4 < 5$, so it cannot be invertible, and so $\mathbf{Ax} = \mathbf{0}$ cannot have an unique solution. It follows that since $\mathbf{Ax} = \mathbf{0}$ does have at least one solution, namely $\mathbf{x} = \mathbf{0}$ (since $\mathbf{A0} = \mathbf{0}$), it must have infinitely many. □

b. Here goes:

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 3 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 4 & 4 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_4 \\ \Longrightarrow \end{matrix} \begin{bmatrix} 1 & 2 & 2 & 4 & 4 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 + R_1 \\ \Longrightarrow \\ R_4 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 2 & 4 & 4 \\ 0 & 2 & 3 & 4 & 3 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & -2 & -2 & -6 & -5 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \Longrightarrow \\ \frac{1}{2}R_3 \\ R_4 + R_2 \end{matrix} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 2 & 3 & 4 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{matrix} \frac{1}{2}R_2 \\ \Longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} & 2 & \frac{3}{2} \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 + R_3 \\ R_2 - \frac{3}{2}R_3 \\ \Longrightarrow \\ R_4 - R_3 \\ R_5 - R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{-\frac{1}{3}R_4}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} R_1 - R_4 \\ R_2 - \frac{1}{2}R_4 \\ R_3 - R_4 \\ \Longrightarrow \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Whew! □

c. Since the reduced matrix has four non-zero rows, $\operatorname{rank}(\mathbf{A}) = 4$; it follows by the Rank-Nullity Law that the nullity of $\mathbf{A}$ is equal to the number of columns of $\mathbf{A}$ minus its rank, *i.e.* $\operatorname{nullity}(\mathbf{A}) = 5 - 4 = 1$. □

d. The column vectors in the original matrix $\mathbf{A}$ corresponding to the columns in which the reduced matrix has leading 1s in non-zero rows form a basis for the column space $\operatorname{col}(\mathbf{A})$:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \\ 1 \end{bmatrix} \right\} \qquad \square$$

e. To find a basis for $\operatorname{null}(\mathbf{A})$, we need to write out the solutions to $\mathbf{Ax} = \mathbf{0}$ in vector-parametric form. (Recall from the solution to part **a** that this homogeneous equation must have infinitely many solutions.) Fortunately, most of the work has already been done in solving part **b** above. If we augment $\mathbf{A}$ with a column of 0s and row-reduce it just as we row-reduced $\mathbf{A}$ in solving part **b**, we'll end up with the reduced matrix from part **b**, augmented with a column of 0s:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

In terms of the entries of $\mathbf{x}$, this comes down to the equations $x_1 + x_5 = 0$, $x_2 - \frac{1}{2}x_5 = 0$, $x_3 = 0$, and $x_4 + x_5 = 0$. Setting $x_5 = t$ for a parameter t and solving the preceding equations in terms of t, we get that the solutions to $\mathbf{Ax} = \mathbf{0}$ are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -t \\ \frac{1}{2}t \\ 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ \frac{1}{2} \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

It follows that $\left\{ \begin{bmatrix} -1 \\ \frac{1}{2} \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for null($\mathbf{A}$). ■

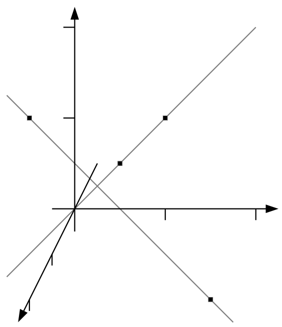
2. Consider the line in $\mathbb{R}^3$ passing through the points $(2,2,0)$ and $(2,0,2)$, and also the line passing through the points $(0,1,1)$ and $(1,1,1)$.

a. Sketch these points and lines. *[2]*

b. Find a parametric description of each of these lines. *[4]*

c. Find the point at which the two lines meet and the (smallest) angle between them at that point. *[4]*

SOLUTIONS. **a.** A sketch:



□

b. The vector from $(2,2,0)$ to $(2,0,2)$ is $\begin{bmatrix} 2-2 \\ 0-2 \\ 2-0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$, so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$ is a parametric description of the first line. Similarly, the vector from $(0,1,1)$ to $(1,1,1)$ is $\begin{bmatrix} 1-0 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a parametrization of the second line. □

c. The two lines meet at a common point exactly when

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix},$$

that is, when $x = 0 + 1t = 2 + 0s$, $y = 1 + 0t = 2 - 2s$, and $z = 1 + 0t = 0 + 2s$. Simplifying, this comes down to $x = t = 2$, $y = 1 = 2 - 2s$, and $z = 1 = 2s$, and it's pretty obvious that this requires $t = 2$ and $s = \frac{1}{2}$. Plugging each value into the parametric description of the appropriate line gives the point $(2,1,1)$.

The angle θ between the lines is the angle between the direction vectors, so

$$\cos(\theta) = \frac{\begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} \right\| \left\| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\|} = \frac{0}{2\sqrt{2} \cdot 1} = 0,$$

so the angle between the lines is $\frac{\pi}{2}$ radians or $90°$, *i.e.* the lines are perpendicular. ■

3. Let $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

a. Find $\mathbf{B}^{-1}$, if it exists. *[10]*
b. Use your work in part **a** to compute $|\mathbf{B}|$. *[5]*

SOLUTIONS. **a.** We set up the superaugmented matrix and Gauss-Jordan away:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_3 - R_1 \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_2 \leftrightarrow R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_4 - R_2 \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_4 - R_3 \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

Since a row of 0s has turned up on the left-hand side of the reduced super-augmented matrix, the matrix $\mathbf{B}$ has no inverse. □

b. Since $\mathbf{B}$ is not invertible by the solution to part **a**, $|\mathbf{B}| = 0$. ■

4. Find an equation of the form $ax + by + cz = d$ for the plane containing both the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and the point $(1, 1, 1)$. *[9]*

SOLUTION. Since every point on the line should be in the plane, the point $(0, 0, 2)$, in particular, must be in the plane. Note also that the direction vector of the line must be parallel to the plane. A second vector parallel to the plane would be the vector that takes one from $(0, 0, 2)$ to $(1, 1, 1)$, namely $\begin{bmatrix} 1-0 \\ 1-0 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. The cross-product of these two vectors will be normal to the plane:

$$\begin{aligned} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} [1 \\ 1 \\ -1 \end{bmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= [1 \cdot (-1) - 1 \cdot 1]\,\mathbf{i} - [(-1) \cdot (-1) - 1 \cdot 1]\,\mathbf{j} + [(-1) \cdot 1 - 1 \cdot 1]\,\mathbf{k} \\ &= -2\mathbf{i} - 0\mathbf{j} - 2\mathbf{k} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \end{aligned}$$

Thus the plane in question has an equation of the form $-2x - 2z = d$. We solve for d by plugging in $(x, y, z) = (0, 0, 2)$: $d = -2 \cdot 0 - 2 \cdot 2 = -4$. It follows that $-2x - 2y = -4$ is an equation of the plane. (If you don't like $-$s or factors of 2, just multiply through by $-\frac{1}{2}$ to get $x+y=2$ ... :-) ■

5. Let $\mathbf{D} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

a. Find all the eigenvalues of $\mathbf{D}$. *[5]*
b. Find all the eigenvectors of $\mathbf{D}$. *[5]*

SOLUTIONS. **a.** First,

$$|\mathbf{D} - \lambda \mathbf{I}_2| = \left| \begin{matrix} 3 & -1 \\ 1 & 1 \end{matrix} \right| - \lambda \left| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right| = \left| \begin{matrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{matrix} \right| = (3-\lambda)(1-\lambda) - 1(-1) = \lambda^2 - 4\lambda + 4\,.$$

Second, $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, which $= 0$ exactly when $\lambda = 2$, so $\mathbf{D}$ has 2 as its only eigenvalue. □

b. To find all the eigenvectors of $\mathbf{D}$ we need to find all solutions $\mathbf{x}$ to $(\mathbf{D} - 2\mathbf{I}_2)\,\mathbf{x} = \mathbf{0}$. We plug $\mathbf{D} - 2\mathbf{I}_2 = \begin{bmatrix} 3-2 & -1 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ into the usual augmented matrix and use the Gauss-Jordan method: $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \underset{R_2 - R_1}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$. It follows that $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector of $\mathbf{D}$ exactly when $x - y = 0$, *i.e.* when $x = y$. That is, the eigenvectors of $\mathbf{D}$ are the scalar multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. ■

Part Z. Do any *three* of **6–11**. *[Subtotal = 36/100]*

6. Use the properties of the vector operations and the dot product to verify that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}\left(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2\right)$. *[12]*

SOLUTION. Recall that $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$ for any vector $\mathbf{x}$. With the help of the distributive and commutative properties of the dot product, it follows that:

$$\begin{aligned} \frac{1}{2}\left(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2\right) &= \frac{1}{2}\left((\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) - \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v}\right) \\ &= \frac{1}{2}\left(\mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v}\right) \\ &= \frac{1}{2}\left(\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u}\right) = \frac{1}{2}\left(2\mathbf{u} \cdot \mathbf{v}\right) = \mathbf{u} \cdot \mathbf{v} \qquad ■ \end{aligned}$$

7. Determine whether $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, |x| = |y| \right\}$ a subspace of $\mathbb{R}^2$ or not. If it is a subspace, determine its dimension. *[12]*

SOLUTION. W is not a subspace of $\mathbb{R}^2$. To see this, observe that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in W$ (since $|1| = 1 = |1|$ and $|1| = 1 = |-1|$, repectively), but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin W$ (since $|2| = 2 \neq 0 = |0|$). As W is not closed under vector addition, it is not a subspace. ■

8. Consider the planes in $\mathbb{R}^3$ given by the equations $2x+2y+z=6$ and $x-y=0$, respectively.

a. Give a parametric description of the line of intersection of these two planes. *[8]*

b. Find the points, if any, in which the line given by $x=t$, $y=3-t$, and $z=1$ intersects each of the two planes. *[4]*

SOLUTIONS. **a.** This boils down to finding the solutions of the system of equations $2x+2y+z=6$ and $x-y=0$. As usual, we set up the augmented matrix and throw the Gauss-Jordan algorithm at it:

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 6 \\ 1 & -1 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & 2 & 1 & 6 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2-2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 4 & 1 & 6 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ \frac{1}{4}R_2 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{3}{2} \end{array}\right] \begin{array}{c} R_1+R_2 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & \frac{3}{2} \\ 0 & 1 & \frac{1}{4} & \frac{3}{2} \end{array}\right] \quad i.e. \quad \begin{array}{c} x+\frac{1}{4}z=\frac{3}{2} \\ y+\frac{1}{4}z=\frac{3}{2} \end{array}$$

Let t be a parameter and set $z=t$; then $x=y=\frac{3}{2}-\frac{1}{4}z$. Thus, the line of intersection of the two planes is given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix} \qquad \square$$

b. We plug the parametric expressions for the given line into each of the equations for the planes and try to solve for the parameter.

First, $2x+2y+z=2t+2(3-t)+1=2t+6-2t+1=7\neq 6$ no matter what value t has, so the line does not intersect the plane $2x+2y+z=6$.

Second, $x-y=t-(3-t)=2t-3=0$ exactly when $t=\frac{3}{2}$. When $t=\frac{3}{2}$, $x=t=\frac{3}{2}$, $y=3-t=3-\frac{3}{2}=\frac{3}{2}$, and $z=1$, so the line intersects the plane $x-y=0$ in the point $\left(\frac{3}{2},\frac{3}{2},1\right)$. ■

9. Find a 2×2 matrix $\mathbf{X}$ such that $\mathbf{X}^2-2\mathbf{X}+\mathbf{I}_2=\mathbf{O}_2$, where $\mathbf{O}_2$ is the 2×2 zero matrix. Is there another such $\mathbf{X}$? Explain why or why not. *[12]*

SOLUTION. It is tempting to treat this like a normal quadratic equation. Giving in (partway) to this temptation, note that $\mathbf{X}^2-2\mathbf{X}+\mathbf{I}_2=(\mathbf{X}-\mathbf{I}_2)^2$ which obviously $=\mathbf{O}_2$ when $\mathbf{X}=\mathbf{I}_2$.

It is also tempting to suppose that this is the only solution, in the same way that $x^2-2x+1=(x-1)^2=0$ only for $x=1$. This temptation should be resisted: the reason $(x-1)^2=(x-1)(x-1)=0$ can only happen if $x-1=0$, *i.e.* $x=1$, is because in the real numbers the only way a^2 can be 0 is if $a=0$. Unfortunately, there are plenty of matrices $\mathbf{A}\neq\mathbf{O}_2$ such that $\mathbf{A}^2=\mathbf{O}_2$. $\mathbf{A}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a simple example of such a matrix. The question is whether there is a matrix $\mathbf{X}\neq\mathbf{I}_2$ (so $\mathbf{X}-\mathbf{I}_2\neq\mathbf{O}_2$) such that $(\mathbf{X}-\mathbf{I}_2)^2=\mathbf{O}_2$. The answer is yes: take any matrix $\mathbf{A}\neq\mathbf{O}_2$ such that $\mathbf{A}^2=\mathbf{O}_2$ and let $\mathbf{X}=\mathbf{A}+\mathbf{I}_2$. Since $\mathbf{A}\neq\mathbf{O}_2$, $\mathbf{X}=\mathbf{A}+\mathbf{I}_2\neq\mathbf{I}_2$, and yet $(\mathbf{X}-\mathbf{I}_2)^2=\mathbf{A}^2=\mathbf{O}_2$. For example, using $\mathbf{A}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{X}=\mathbf{A}+\mathbf{I}_2=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is another solution to $\mathbf{X}^2-2\mathbf{X}+\mathbf{I}_2=(\mathbf{X}-\mathbf{I}_2)^2=\mathbf{O}_2$. ■

10. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that

$$T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix},\ T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix},\ \text{and } T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

a. Find $\begin{bmatrix}x\\y\\z\end{bmatrix}$ such that $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}2\\1\\0\end{bmatrix}$. *[4]* **b.** Compute $T\left(\begin{bmatrix}3\\4\\5\end{bmatrix}\right)$. *[8]*

SOLUTION. **a.** We reverse-engineer the desired vector with the help of the linearity of T:

$$\begin{bmatrix}2\\1\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} = 2T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right) = T\left(2\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\1\\2\end{bmatrix}\right) = T\left(\begin{bmatrix}2\\5\\8\end{bmatrix}\right)$$

Thus $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\5\\8\end{bmatrix}$ does the job. $\square$

b. We will use the linearity of T again, but to do so we need to find scalars a, b, and c such that $a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}0\\1\\2\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}3\\4\\5\end{bmatrix}$. Boringly, but as usual, we set up the augmented matrix and use the Gauss-Jordan algorithm:

$$\left[\begin{array}{ccc|c}1&0&0&3\\2&1&0&4\\3&2&1&5\end{array}\right] \begin{array}{c}\Longrightarrow\\ R_2-2R_1\\ R_3-3R_1\end{array} \left[\begin{array}{ccc|c}1&0&0&3\\0&1&0&-2\\0&2&1&-4\end{array}\right] \begin{array}{c}\Longrightarrow\\ R_3-2R_2\end{array} \left[\begin{array}{ccc|c}1&0&0&3\\0&1&0&-2\\0&0&1&0\end{array}\right]$$

It follows that $\begin{bmatrix}3\\4\\5\end{bmatrix} = 3\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}0\\1\\2\end{bmatrix}$, so

$$\begin{aligned} T\left(\begin{bmatrix}3\\4\\5\end{bmatrix}\right) &= T\left(3\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}0\\1\\2\end{bmatrix}\right) = 3T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) - 2T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right) \\ &= 3\begin{bmatrix}1\\0\\0\end{bmatrix} - 2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}3\\-2\\0\end{bmatrix}. \qquad \blacksquare \end{aligned}$$

11. Find an orthogonal basis for $U = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. *[12]*

SOLUTION. First, we need to find a basis for U. We assemble the vectors in the spanning set into the columns of a matrix and reduce it:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \overset{R_2 - R_1}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\underset{R_3 - R_2}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \underset{R_4 - R_3}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns of the original matrix corresponding to the columns in the reduced matrix in which a leading 1 of a row occurs give a basis for the column space. It follows that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for U.

We now orthogonalize this basis using the Gram-Schmidt process:

We take the first basis vector unchanged: $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Then

$$\mathbf{b_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix},$$

and

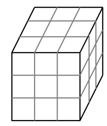
$$\mathbf{b_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{0}{1} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\frac{3}{2}} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}.$$

It follows that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for U. ■

[Total = 100]

Part ♡. Bonus!



••. A dangerously sharp tool is used to cut a cube with a side length of $3\,cm$ into 27 smaller cubes with a side length of $1\,cm$. This can be done easily with six cuts. Can it be done with fewer? (You may rearrange the pieces between cuts.) If so, explain how; if not, explain why not. *[1]*

SOLUTION. It cannot be done with less than six cuts. They key is to consider the small cube that is completely inside (that is, no face of it is part of a face of) the original cube. Each of the six faces of this smaller cube must have come from a different cut. ■

∘∘. Write an original little poem about linear algebra or mathematics in general. *[2]*

SOLUTION. You're on your own on this one ... ■

ENJOY THE REST OF THE SUMMER!