Mathematics 1350H – Linear algebra I: Matrix algebra
TRENT UNIVERSITY, Summer 2013

Assignment #4
Due on Wednesday, 12 June, 2013.

Determinants the Gauss-Jordan way

Given a square matrix $\mathbf{A}$, we can compute a number called the *determinant* of $\mathbf{A}$, usually denoted by $|\mathbf{A}|$ or $\det(\mathbf{A})$, that gives a lot of information about $\mathbf{A}$. For example, $|\mathbf{A}| \neq 0$ exactly when $\mathbf{A}^{-1}$ exists. One problem with the usual definition of determinants – which works by reducing the determinant of an $n \times n$ matrix to a weighted sum of n determinants of $(n-1) \times (n-1)$ matrices - is that computing them this way is a *lot* of work unless $\mathbf{A}$ is a pretty small matrix. (Heck, it's a pain even for 3×3 matrices with the usual definition ...) Here are some facts which let you compute the determinant of a matrix using the Gauss-Jordan method:

The determinant of an $n \times n$ matrix $\mathbf{A}$ satisfies the following rules:

i. The identity matrix has determinant equal to 1, *i.e.* $|\mathbf{I}_n| = 1$.

ii. If you exchange the ith and jth row of $\mathbf{A}$ to get the matrix $\mathbf{B}$, then $|\mathbf{B}| = -|\mathbf{A}|$.

iii. If you multiply the ith row of $\mathbf{A}$ by a constant c to get the matrix $\mathbf{C}$, then $|\mathbf{C}| = c|\mathbf{A}|$.

iv. If you add a multiple of any row of $\mathbf{A}$ to a different row of $\mathbf{A}$ to get the matrix $\mathbf{D}$, then $|\mathbf{D}| = |\mathbf{A}|$. (In general, if you add any row vector $\mathbf{r}$ to the ith row of $\mathbf{A}$ to get the matrix $\mathbf{D}$, then $|\mathbf{D}| = |\mathbf{A}| + |\mathbf{A}_{i,\mathbf{r}}|$, where $\mathbf{A}_{i,\mathbf{r}}$ is the matrix $\mathbf{A}$ with its ith row replaced by $\mathbf{r}$.)

v. Taking the transpose of $\mathbf{A}$ doesn't change the determinant. That is, $|\mathbf{A}^T| = |\mathbf{A}|$.

If you really wanted to, by the way, you could actually use this collection of rules as the definition of the determinant of a matrix. It's pretty cumbersome as a definition, but it does provide a much more efficient way to compute the determinant of even a modestly large matrix.

1. Use rules *i* – *v*, as well as **1** and **2**, to compute $|\mathbf{A}|$ if:

a. $\mathbf{A}$ has a column or a row of zeros. *[1.5]*

b. $\mathbf{A}$ has two equal columns or two equal rows. *[1.5]*

c. $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$. *[2]*

2. Rules *ii* – *iv* are true for the columns of $\mathbf{A}$ as well as the rows. Why? *[2]*

3. Use the Gauss-Jordan method to put the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 0 \end{bmatrix}$ in reduced row-echelon form. Apply what you have learned above to use this computation to determine $|\mathbf{A}|$. *[3]*

Bonus. Assuming the general part of rule *iv* (the part in parentheses) is true, show that the particular part of rule *iv* (the part not in parentheses) must be true. You may use the other rules as well. *[2]*