Trent University

MATH 1350H Test

4 November, 2009

Time: 50 minutes

Name: Solutions

Student Number: 00000000

Question	Mark
1	____
2	____
3	____
4	____
Total	____

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet (or an annotated *Formula for Success*).

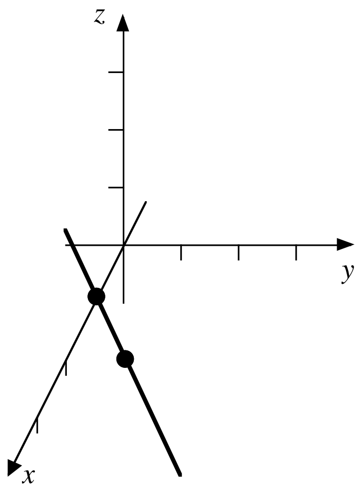
1. Consider the line passing through the points $(1,0,0)$ and $(2,1,0)$.

a. Sketch this line. *[2]*

b. Find a parametric description of this line. *[4]*

c. What is the acute angle between this line and the plane given by $y+z=1$? *[4]*

Solution to **a**. We plot the two points and draw the line joining them:



Note that the line is entirely in the xy plane. ■

Solution to **b**. We'll use $(1,0,0)$ as the base point, and the vector from $(1,0,0)$ to $(2,1,0)$, $\begin{bmatrix} 2-1 \\ 1-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, as the direction vector. This gives $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ as a parametric description of the line. ■

Solution to **c**. We first compute the angle α between the direction vector of the line and the normal vector of the plane:

$$\cos(\alpha) = \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\| \left\| \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\|} = \frac{1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1}{\sqrt{1^2+1^2+0^2}\sqrt{0^2+1^2+1^2}} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

Thus $\alpha = 60° = \frac{\pi}{3}$ *rad*.

However, we really want the angle between the (direction vector of the) line and the plane, which will be $90° - \alpha = 30°$ or $\frac{\pi}{2} - \alpha = \frac{\pi}{6}$ *rad*. ■

2. Consider the following system of linear equations:
$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 6 \\ 2x & - & y & + & z & = & 3 \\ 3x & + & y & - & z & = & 2 \end{array}$$

a. Find the solution(s), if any, of this system of equations. *[7]*

b. What does your answer to **a** tell you about some planes? *[1.5]*

c. What does your answer to **a** tell you about some vectors? *[1.5]*

SOLUTION TO **a**. We set up the augmented matrix and reduce it the Gauss-Jordan way:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 3 & 1 & -1 & 2 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & -2 & -4 & -16 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 \leftrightarrow R_3 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & -4 & -16 \\ 0 & -3 & -1 & -9 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ -\frac{1}{2}R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & -3 & -1 & -9 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \Longrightarrow \\ R_3 + 3R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 5 & 15 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ \frac{1}{5}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} R_1 + R_3 \\ R_2 - 2R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This tells that there is just one solution to the given system of linear equations: $x = 1$, $y = 2$, and $z = 3$. ∎

SOLUTION TO **b**. The answer to **a** tells us that the three planes given by the linear equations $x + y + z = 6$, $2x - y + z = 3$, and $3x + y - z = 2$, respectively, intersect in a single point, $(1, 2, 3)$. ∎

SOLUTION TO **c**. The answer to **a** tells us that the vector $\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$ is a linear combination (*i.e.* is in the span of) the three vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, namely

$$1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}. \qquad ∎$$

3. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. Find a linear equation for the plane given by the vector-parametric equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

b. Sketch the plane $x + 2y + 3z = 6$.

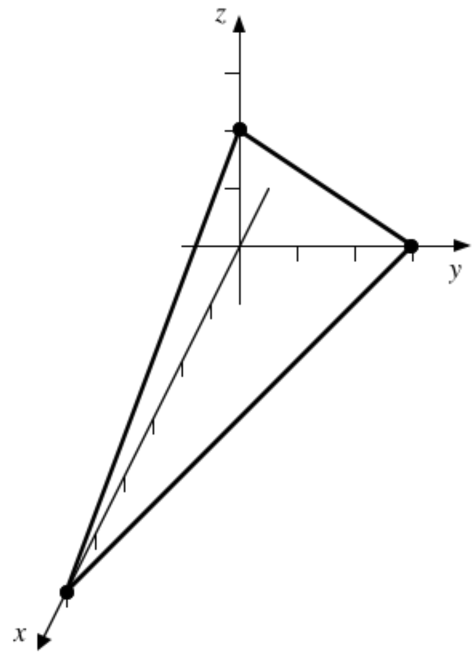
c. Find the shortest distance from the point $(1, 1, 2)$ to the plane $x + y + z = 1$.

Solution to **a**. We will need a normal vector for the plane, which we will obtain by taking the cross-product of the given direction vectors.

$$\begin{aligned} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\ &= (1 \cdot 1 - 0 \cdot 1)\mathbf{i} - ((-1) \cdot 1 - 0 \cdot 0)\,\mathbf{j} + ((-1) \cdot 1 - 0 \cdot 1)\,\mathbf{k} \\ &= \mathbf{i} + \mathbf{j} - \mathbf{k} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

It follows that the plane has an equation of the form $x + y - z = d$. To determine d, we plug in a point of the plane; one such is given by the base vector , $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, of the parametric description. Thus $d = 1 + 1 - 1 = 1$, and so one linear equation for the given plane is $x + y - z = 1$. ■

Solution to **b**. We first find the intercepts of the given plane. Plugging $y = z = 0$ into $x + 2y + 3z = 6$ and solving for x gives us $x = 6$, so the x-intercept of the plane is $(6, 0, 0)$. Similarly, plugging in $x = z = 0$ gives $y = 3$, so the y-intercept is $(0, 3, 0)$, and plugging in $x = y = 0$ gives $z = 2$, so the z-intercept is $(0, 0, 2)$. Now we plot these three points and join them up; the resulting triangle is the part of the given plane that is in the first octant.



■

Solution to **c**. We first find a point on the plane. Setting $y = z = 0$ and solving for x in $x + y + z = 1$ gives us the point $(1, 0, 0)$. The vector from this point to the given point is $\mathbf{v} = \begin{bmatrix} 1-1 \\ 1-0 \\ 2-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. We project this vector onto the normal vector for the plane, $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$:

$$\mathrm{proj}_{\mathbf{n}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} = \frac{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{0 \cdot 1 + 1 \cdot 1 + 1 \cdot 2}{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The distance from the point to the plane is the length of this projection: $\|\mathrm{proj}_{\mathbf{n}}(\mathbf{v})\| = \left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. ■

4. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. Why isn't every vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$?

b. Compute $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^8 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

c. Find a 2×3 matrix $\mathbf{A}$ such that $\mathbf{A}\mathbf{A}^T = \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution to **a**. Because in three dimensions you need at least three vectors to be able to span everything … ■

Solution to **b**. Here goes:

$$\begin{aligned}
\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^8 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix} \qquad \blacksquare
\end{aligned}$$

Solution to **c**. Our inspiration here is that $\mathbf{I}_2^T = \mathbf{I}_2$ and so $\mathbf{I}_2 \mathbf{I}_2^T = \mathbf{I}_2$. We pad out $\mathbf{I}_2$ with an extra column of 0s to get our matrix: $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$\mathbf{A}\mathbf{A}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2 \,,$$

so the job is done! ■

[Total = 40]