Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2009

Solutions to Assignment #6
A t0y un1verse

We're going to define our vectors using a different bunch of scalars, namely $\mathbb{Z}_2 = \{0, 1\}$, where $+$ and $\cdot$ are given by the following tables:

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

If you know about modular arithmetic, this is just addition and multiplication modulo 2. You may take it on faith that this gives something algebraically well-behaved enough to be usable as a set of scalars.

The set of three-dimensional vectors we get from these scalars is

$$\mathbb{Z}_2^3 = \left\{ \begin{bmatrix} u \\ v \\ w \end{bmatrix} \,\middle|\, \text{each of } u,\ v,\ w \text{ is 0 or 1} \right\},$$

with addition of vectors and multiplication by scalars given by

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} a+r \\ b+s \\ c+t \end{bmatrix} \qquad \text{and} \qquad \alpha \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \alpha \cdot u \\ \alpha \cdot v \\ \alpha \cdot w \end{bmatrix},$$

using the addition and multiplication of scalars defined above.

1. How many vectors are there in $\mathbb{Z}_2^3$? List them all! *[2]*

SOLUTION. There are eight vectors in $\mathbb{Z}_2^3$, namely

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{and } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \blacksquare$$

2. If $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a vector of $\mathbb{Z}_2^3$, what is $-\mathbf{u}$? Why? *[2]*

SOLUTION. $-\mathbf{u} = \mathbf{u}$ for any vector $\mathbf{u}$ in $\mathbb{Z}_2^3$. The reason is that $1 + 1 = 0$ in $\mathbb{Z}_2$, and so $-1 = 1$ in $\mathbb{Z}_2$. Thus $-\mathbf{u} = (-1)\mathbf{u} = 1\mathbf{u} = \mathbf{u}$. $\blacksquare$

3. How many subspaces does $\mathbb{Z}_2^3$ have? List them all! *[4]*

SOLUTION. $\mathbb{Z}_2^3$ has sixteen (16) subspaces: one 0-dimensional subspace,

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

seven 1-dimensional subspaces,

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\},$$
$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \text{ and } \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

seven 2-dimensional subspaces,

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\},$$
$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\},$$
$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$
$$\text{and } \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

and one 3-dimensional subspace, namely

$$\mathbb{Z}_2^3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

itself. ■

4. Find an example of weird behaviour by the dot product of vectors in $\mathbb{Z}_2^3$. *[2]*

SOLUTION. One nasty problem is illustrated by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0 = 1 + 1 + 0 = (1 + 1) + 0 = 1 + 1 = 0\,.$$

If the dot product worked out in $\mathbb{Z}_2^3$ as it does in $\mathbb{R}^3$, this would mean that the vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is orthogonal to itself and has length 0, even though it isn't the zero vector. (Is this weird enough? :-) ■