Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2009

SOLUTIONS TO ASSIGNMENT #2

Objection to projection is a basis for dejection!

The key to what follows is the following idea. Recall (from class and §1.2) that the component of a vector $\mathbf{v}$ parallel to a (non-zero) vector $\mathbf{u}$ is the *projection of* $\mathbf{v}$ *onto* $\mathbf{u}$*:*

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u}$$

Recall further that if you take away the component of $\mathbf{v}$ which is parallel to $\mathbf{u}$ away from $\mathbf{v}$, the component that is left is orthogonal to $\mathbf{u}$.

1. Suppose $\mathbf{v}$ and $\mathbf{u} \neq \mathbf{0}$ are vectors of the same dimension. Verify that $\mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ is orthogonal to $\mathbf{u}$. *[2]*

Hint: Use the dot product!

SOLUTION. Following the hint, we will verify that $\mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ is orthogonal to $\mathbf{u}$ by taking the dot product of the two vectors:

$$(\mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})) \cdot \mathbf{u} = \left(\mathbf{v} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u}\right) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} = 0$$

(The last bit works since the dot product is commutative.)

Since their dot product is zero, $\mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ is orthogonal to $\mathbf{u}$. ■

Now let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ be a set of four vectors in 4-dimensional space. We will modify this set of vectors to make it nicer in some respects.

2. Use the idea in **1** to modify the second vector in S to make it orthogonal to the first vector in S. *[2]*

SOLUTION. We will apply the formula in **1** with $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ to get a new vector $\mathbf{v}'$ which is orthogonal to $\mathbf{u}$.

$$\mathbf{v}' = \mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \left(\frac{\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}}\right)\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{1\cdot 1 + 1\cdot 1 + (-1)\cdot 1 + 1\cdot 1}{1\cdot 1 + 1\cdot 1 + (-1)\cdot(-1) + 1\cdot 1}\begin{bmatrix}1\\1\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\1\\-1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{3}{2}\\\frac{1}{2}\end{bmatrix} \quad \blacksquare$$

3. Modify the third vector in S to make it orthogonal to both the first and second vectors in S. *[2]*

SOLUTION. We will adapt the formula in **1** with $\mathbf{u} = \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}$, $\mathbf{v}' = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{3}{2}\\\frac{1}{2}\end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$ to get a new vector $\mathbf{w}'$ which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}'$.

$$\mathbf{w}' = \mathbf{w} - \mathrm{proj}_{\mathbf{u}}(\mathbf{w}) - \mathrm{proj}_{\mathbf{v}'}(\mathbf{w}) = \mathbf{w} - \left(\frac{\mathbf{u}\cdot\mathbf{w}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} - \left(\frac{\mathbf{v}'\cdot\mathbf{w}}{\mathbf{v}'\cdot\mathbf{v}'}\right)\mathbf{v}'$$

$$= \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \left(\frac{\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}}\right)\begin{bmatrix}1\\1\\-1\\1\end{bmatrix} - \left(\frac{\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{3}{2}\\\frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\\1\end{bmatrix}}{\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{3}{2}\\\frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{3}{2}\\\frac{1}{2}\end{bmatrix}}\right)\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{3}{2}\\\frac{1}{2}\end{bmatrix}$$

(Skipping some arithmetic . . . :-)

$$= \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{0}{4}\begin{bmatrix}1\\1\\-1\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{3}{2}\\\frac{1}{2}\end{bmatrix} = \begin{bmatrix}-\frac{1}{3}\\-\frac{1}{3}\\0\\\frac{2}{3}\end{bmatrix}$$

We leave it ti the reader to check that $\mathbf{w}'$ is indeed orthogonal to $\mathbf{u}$ and $\mathbf{v}'$. (As well as the original $\mathbf{v}$!) $\blacksquare$

4. Modify the fourth vector in S to make it orthogonal to all of the first three vectors in S. *[1]*

SOLUTION. We will adapt the formula in **1** with $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}' = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$, $\mathbf{w}' = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ to get a new vector $\mathbf{x}'$ which is orthogonal to $\mathbf{u}$, $\mathbf{v}'$, and $\mathbf{w}'$.

$$\begin{aligned}
\mathbf{x}' &= \mathbf{x} - \mathrm{proj}_{\mathbf{u}}(\mathbf{x}) - \mathrm{proj}_{\mathbf{v}'}(\mathbf{x}) - \mathrm{proj}_{\mathbf{w}'}(\mathbf{x}) \\
&= \mathbf{x} - \left(\frac{\mathbf{u}\cdot\mathbf{x}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} - \left(\frac{\mathbf{v}'\cdot\mathbf{x}}{\mathbf{v}'\cdot\mathbf{v}'}\right)\mathbf{v}' - \left(\frac{\mathbf{w}'\cdot\mathbf{x}}{\mathbf{w}'\cdot\mathbf{w}'}\right)\mathbf{w}' \\
&= \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}}\right)\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}\cdot\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}}\right)\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} \\
&\qquad - \left(\frac{\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}\cdot\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}}\right)\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix} \\
&\qquad \text{(Skipping more arithmetic \dots :-)} \\
&= \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \frac{0}{4}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} - \frac{\frac{2}{3}}{\frac{2}{3}}\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We leave it ti the reader to check that $\mathbf{x}'$ is indeed orthogonal to $\mathbf{u}$, $\mathbf{v}'$, and $\mathbf{w}'$. (As well as the original $\mathbf{v}$ and $\mathbf{w}$!) ■

NOTE A. It is important in the solutions to questions **3** and **4** given above to work with the modified vectors $\mathbf{v}'$ and, in **4**, $\mathbf{w}'$. Working with the unmodified $\mathbf{v}$ and $\mathbf{w}$ would require correcting for the fact that these might not be orthogonal to $\mathbf{u}$ (though $\mathbf{w}$ actually is), or to each other. The difficulty is that if that they are not orthogonal, projections onto them will overlap and interfere with each other.

5. Further modify all of your modified vectors from **2**–**4** to have length one. *[1]*

SOLUTION. We will multiply each of $\mathbf{u}$, $\mathbf{v}'$, $\mathbf{w}'$, and $\mathbf{x}'$, in the notation of the solution above, by the reciprocal of its length to get new vectors of length one that we'll call $\mathbf{a}$, $\mathbf{b}$,

c, and **d**, respectively. Note that we can reuse some numbers obtained in the solutions to questions **2** through **4**.

$$\mathbf{a} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{\mathbf{u}\cdot\mathbf{u}}}\mathbf{u} = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{b} = \frac{1}{\|\mathbf{v}'\|}\mathbf{v}' = \frac{1}{\sqrt{\mathbf{v}'\cdot\mathbf{v}'}}\mathbf{v}' = \frac{1}{\sqrt{3}}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\mathbf{c} = \frac{1}{\|\mathbf{w}'\|}\mathbf{w}' = \frac{1}{\sqrt{\mathbf{w}'\cdot\mathbf{w}'}}\mathbf{w}' = \frac{1}{\sqrt{\frac{2}{3}}}\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix} = \frac{\sqrt{3}}{\sqrt{2}}\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}$$

$$\mathbf{d} = \frac{1}{\|\mathbf{x}'\|}\mathbf{x}' = \frac{1}{\sqrt{\mathbf{x}'\cdot\mathbf{x}'}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(Omitting a bit of \ldots :-)}$$

Those who want to do more arithmetic may take the multipliers into the vectors, coordinate by coordinate, or collect common factors of the coordinates into the multipliers. ■

NOTE B. The procedure used in the solutions to **2**–**5** is the Gram-Schmidt process for modifying a set of basis vectors into an *orthonormal* set of basis vectors. It is described in detail in Chapter 5 of the text.

6. What might your final collection of modified vectors from **5** be good for? *[2]*

SOLUTION. The vectors **a**, **b**, **c**, and **d** are unit vectors in four-dimensional space which are mutually perpendicular. Like the vectors **i**, **j**, and **k** in three-dimensional space, they can serve as the basis for a coordinate system: they provide directions for the axes, along with a reference length of one. ■