Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2009

FINAL EXAMINATION
Tuesday, 22 December, 2009

Time: 3 hours *Brought to you by* Стефан Біланюк.
Instructions: Show all your work. *If in doubt about something,* **ask!**
Aids: Calculator; annotated *Formula for Success* or $8.5'' \times 11''$ aid sheet; $\leq 10^{11}$ neurons.

Part I. Do all of **1**–**5**. *[Subtotal = 64/100]*

1. Consider the planes in $\mathbb{R}^3$ given by the equations $x + y + \sqrt{2}z = 6$ and $z = 0$.

a. Sketch these planes. *[4]*
b. Find the angle between these planes. *[3]*
c. Find a parametric description of the line in which these planes intersect. *[3]*

2. Consider the following system of linear equations and its coefficient matrix $\mathbf{A}$:

$$\begin{array}{rcrcrcrcrcr} x & - & y & + & z & - & u & + & v & = & 1 \\ x & + & 2y & + & 4z & + & 2u & + & v & = & 10 \\ & & 3y & + & 3z & + & 3u & & & = & 9 \\ & & & & z & + & u & + & v & = & 3 \end{array} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 4 & 2 & 1 \\ 0 & 3 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Note that $x = y = z = u = v = 1$ satisfies all four equations.

a. Without any calculation, does this system have no solutions, just one solution, or many solutions? Explain why. *[2]*
b. Use the Gauss-Jordan method to find all the solutions, if any, of this system. *[10]*
c. What are the rank and nullity of $\mathbf{A}$? *[1]*
d. Without any further calculation, give a basis for the column space of $\mathbf{A}$. *[3]*
e. Without any further calculation, give a basis for the null space of $\mathbf{A}$, and explain why it is one. *[4]*

3. Let $\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

a. Find all the eigenvalues of $\mathbf{B}$. *[8]*
b. Find all the eigenvectors of $\mathbf{B}$ corresponding to the eigenvalue $\lambda = 1$. *[7]*

4. Find the inverse matrix, if it exists, of $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. *[10]*

5. Find the shortest distance from the point $(3, 3, 1)$ to the line given by the parametric equations $x = 2$, $y = t$, and $z = 2 + t$. *[9]*

*[Parts **II** and $\odot$ are on page 2.]*

Part II. Do any *three* of **6–11**. *[Subtotal = 36/100]*

6. Use the properties of the vector operations and the dot product to verify that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^2$, then $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$. *[12]*

7. Is $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle|\ xy = 0 \right\}$ a subspace of $\mathbb{R}^2$ or not? Explain why or why not. If it is a subspace, what is its dimension? *[12]*

8. Find a linear equation giving the plane that is described by the parametric equations $x = 1 + 2t$, $y = 2 + s + t$, and $z = 3 + 2s$, where s and t are the parameters. *[12]*

9. Suppose $\mathbf{A}$ is an $n \times n$ matrix such that $\mathbf{Ax} = \mathbf{b}$ has a unique solution $\mathbf{x}$. Does the matrix equation $(\mathbf{XA})^T\mathbf{A} = \mathbf{A} + 3\mathbf{I}_n$ have a unique solution, too? Solve it if it does and explain why it does; if not, explain why it doesn't. *[12]*

10. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is an invertible linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ T\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \text{and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Find the associated matrix $\left[T^{-1}\right] = \mathbf{A}_{T^{-1}}$ of T^{-1}. *[12]*

11. Determine whether $\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$ is in $S = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \right\}$ or not, and find a basis for S. *[12]*

[Total = 100]

Part $\odot$. Bonus! *The problem in this part goes round and round!*

4π. Write an original little poem about linear algebra or mathematics in general. *[2]*

HAVE A GREAT BREAK!
SEE YOU NEXT TERM, I HOPE!