Mathematics 1350H – Linear algebra I: matrix algebra
Trent University, Fall 2009
Solutions to the Final Examination

Time: 3 hours *Brought to you by* Стефан Біланюк.
Instructions: Show all your work. *If in doubt about something,* **ask!**
Aids: Calculator; annotated *Formula for Success* or $8.5'' \times 11''$ aid sheet; $\leq 10^{11}$ neurons.

Part I. Do all of **1**–**5**. *[Subtotal = 64/100]*

1. Consider the planes in $\mathbb{R}^3$ given by the equations $x + y + \sqrt{2}z = 6$ and $z = 0$.

a. Sketch these planes. *[4]*

Solution. ■

b. Find the angle between these planes. *[3]*

Solution. The angle between the planes is the angle between their normal vectors, which we can read off the coefficients of their equations: $x + y + \sqrt{2}z = 6$ has normal vector $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$ and $z = 0x + 0y + z = 0$ has normal vector $\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. If θ is the angle between these vectors, and hence between the planes, then

$$\cos(\theta) = \frac{\mathbf{n} \cdot \mathbf{k}}{\|\mathbf{n}\|\|\mathbf{k}\|} = \frac{1 \cdot 0 + 1 \cdot 0 + \sqrt{2} \cdot 1}{\sqrt{1^2 + 1^2 + \sqrt{2}^2}\sqrt{0^2 + 0^2 + 1^2}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}},$$

so $\theta = 45^\circ = \pi/2$ rad. ■

c. Find a parametric description of the line in which these planes intersect. *[3]*

Solution. ■

2. Consider the following system of linear equations and its coefficient matrix $\mathbf{A}$:

$$\begin{array}{rcrcrcrcrcr} x & - & y & + & z & - & u & + & v & = & 1 \\ x & + & 2y & + & 4z & + & 2u & + & v & = & 10 \\ & & 3y & + & 3z & + & 3u & & & = & 9 \\ & & & & z & + & u & + & v & = & 3 \end{array} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 4 & 2 & 1 \\ 0 & 3 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Note that $x = y = z = u = v = 1$ satisfies all four equations.

a. Without any calculation, does this system have no solutions, just one solution, or many solutions? Explain why. *[2]*

Solution. This system has many solutions. First, it has at least one solution because $x = y = z = u = v = 1$ is a solution. Second, since the system has five variables and only four equations, it cannot have just one solution. ■

b. Use the Gauss-Jordan method to find all the solutions, if any, of this system. *[10]*

SOLUTION. We set up the "super-augmented" matrix for the system and apply the Gauss-Jordan algorithm:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 2 & 4 & 2 & 1 & 10 \\ 0 & 3 & 3 & 3 & 0 & 9 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] \begin{array}{c} R_2 - R_1 \\ \Longrightarrow \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 3 & 3 & 3 & 0 & 9 \\ 0 & 3 & 3 & 3 & 0 & 9 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} \frac{1}{3}R_2 \\ \Longrightarrow \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 3 & 3 & 3 & 0 & 9 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ \Longrightarrow \\ R_3 - 3R_2 \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 4 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_3 \leftrightarrow R_4 \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 4 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - 2R_3 \\ R_2 - R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & -1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

It remains to interpret this result. The reduced matrix represents the linear system $x - 2u - v = -2$, $y - v = 0$, and $z + u + v = 3$, which has the same solutions as the original system. We can easily solve this system for x, y, and z in terms of u and v, so we set $u = s$ and $v = t$ for parameters s and t to get the parametric representation of the set of solutions: $x = -2 + u + v = -2 + s + t$, $y = v = t$, $z = 3 - u - v = 3 - s - t$, $u = s$, and $v = t$. The vector-parametric form can be read off from these equations, coordinate by coordinate:

$$\begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad \blacksquare$$

c. What are the rank and nullity of **A**? *[1]*

SOLUTION. The reduced coefficient matrix in the solution to **b** has three non-zero rows, so the rank of **A** is 3. Since **A** has 5 columns, it follows by the rank-nullity law that the nullity of **A** is $5 - 3 = 2$. ■

d. Without any further calculation, give a basis for the column space of **A**. *[3]*

SOLUTION. The reduced coefficient matrix in the solution to **b** has leading 1s in its three non-zero rows in columns 1, 2, and 3. The corresponding columns of the original matrix **A** form a basis for the column space of **A**, namely:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \\ 1 \end{bmatrix} \right\} \qquad \blacksquare$$

e. Without any further calculation, give a basis for the null space of $\mathbf{A}$, and explain why it is one. *[4]*

SOLUTION. If we had done the reduction in **b** with a right-hand side of all 0s – *i.e.* done the calculation to find the null space of $\mathbf{A}$ – we would have reached the same reduced matrix, except with a right-hand side column of all 0s, resulting in the parametric solution

$$\begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} .$$

The vectors corresponding to each parameter then form a basis for the null space of $\mathbf{A}$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \blacksquare$$

3. Let $\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

a. Find all the eigenvalues of $\mathbf{B}$. *[5]*

SOLUTION. First,

$$\mathbf{B} - \lambda \mathbf{I} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{bmatrix} .$$

Second,

$$\begin{aligned} |\mathbf{B} - \lambda \mathbf{I}| &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix} \qquad \text{[Now we expand along the first column.]} \\ &= (1-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} - 0 \begin{vmatrix} 0 & 1 \\ 2 & 1-\lambda \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 1-\lambda & 2 \end{vmatrix} \\ &= (1-\lambda)\left((1-\lambda)(1-\lambda) - 2 \cdot 2\right) - 0 + 0 \\ &= (1-\lambda)\left(1 - 2\lambda + \lambda^2 - 4\right) \\ &= (1-\lambda)\left(\lambda^2 - 2\lambda - 3\right) \\ &= (1-\lambda)(\lambda - 3)(\lambda + 1) \end{aligned}$$

Other means failing, one could use the quadratic formula at the last step: the roots of $\lambda^2 - 2\lambda - 3$ are, according to the formula,

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = 1 \pm 2 = 3 \text{ or } -1\,;$$

that is $\lambda^2 - 2\lambda - 3 = (\lambda - 3)\,(\lambda - (-1)) = (\lambda - 3)(\lambda + 1)$.

Third, setting $|\mathbf{B} - \lambda\mathbf{I}| = (1 - \lambda)(\lambda - 3)(\lambda + 1) = 0$, it is clear that the eigenvalues of $\mathbf{B}$ are $\lambda = 1$, 3, and -1. ■

b. Find all the eigenvectors of $\mathbf{B}$ corresponding to the eigenvalue $\lambda = 1$. *[5]*

SOLUTION. We need to find all the vectors $\mathbf{u}$ such that $(\mathbf{B} - 1\mathbf{I})\mathbf{x} = \mathbf{0}$, *i.e.* all the solutions to

$$\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We set up the corresponding augmented matrix and apply the Gauss-Jordan algorithm:

$$\left[\begin{array}{ccc||c} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \begin{array}{c} \Longrightarrow \\ \frac{1}{2}R_2 \\ \frac{1}{2}R_3 \end{array} \left[\begin{array}{ccc||c} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 \leftrightarrow R_3 \\ \Longrightarrow \\ \ \end{array} \left[\begin{array}{ccc||c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} \ \\ \Longrightarrow \\ R_3 - R_2 \end{array} \left[\begin{array}{ccc||c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This corresponds to the linear system $y = 0$ and $z = 0$; note that x can be anything, so we set it equal to the parameter t. Thus all the eigenvectors of $\mathbf{B}$ corresponding to the eigenvalue $\lambda = 1$ are given parametrically by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ where } t \in \mathbb{R}. \ ■.$$

4. Find the inverse matrix, if it exists, of $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. *[10]*

SOLUTION. We set up the "super-augmented" matrix $[\,\mathbf{C} \mid \mathbf{I}\,]$ and attempt to reduce it to $\left[\,\mathbf{I} \mid \mathbf{C}^{-1}\,\right]$.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_3 - R_1 \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_2 \leftrightarrow R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ \ \\ R_4 - R_3 \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

It follows that $\mathbf{C}$ is invertible and $\mathbf{C}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$. ■

5. Find the shortest distance from the point $(3, 3, 1)$ to the line given by the parametric equations $x = 2$, $y = t$, and $z = 2 + t$. *[9]*

SOLUTION. In vector-parametric form, the line is given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, so $(2, 0, 2)$ is a point on the line and $\mathbf{d} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is a direction vector for the line. The vector from $(2, 0, 2)$ to $(3, 3, 1)$ is $\mathbf{u} = \begin{bmatrix} 3-2 \\ 3-0 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$. The distance from the point $(3, 3, 1)$ to the given line is the length of the component of $\mathbf{u}$ which is perpendicular to $\mathbf{d}$, that is, $\|\mathbf{u} - \mathrm{proj}_{\mathbf{d}}(\mathbf{u})\|$. ■

Part II. Do any *three* of **6–11**. *[Subtotal = 36/100]*

6. Use the properties of the vector operations and the dot product to verify that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^2$, then $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$. *[12]*

7. Is $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; xy = 0 \right\}$ a subspace of $\mathbb{R}^2$ or not? Explain why or why not. If it is a subspace, what is its dimension? *[12]*

8. Find a linear equation giving the plane that is described by the parametric equations $x = 1 + 2t$, $y = 2 + s + t$, and $z = 3 + 2s$, where s and t are the parameters. *[12]*

9. Suppose $\mathbf{A}$ is an $n \times n$ matrix such that $\mathbf{Ax} = \mathbf{b}$ has a unique solution $\mathbf{x}$. Does the matrix equation $(\mathbf{XA})^T\mathbf{A} = \mathbf{A} + 3\mathbf{I}_n$ have a unique solution, too? Solve it if it does and explain why it does; if not, explain why it doesn't. *[12]*

10. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is an invertible linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; T\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Find the associated matrix $\left[T^{-1}\right] = \mathbf{A}_{T^{-1}}$ of T^{-1}. *[12]*

11. Determine whether $\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$ is in $S = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \right\}$ or not, and find a basis for S. *[12]*

[Total = 100]

Part $\odot$. Bonus! *The problem in this part goes round and round!*

4π. Write an original little poem about linear algebra or mathematics in general. *[2]*

Have a great break!
See you next term, I hope!