Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2008

Solutions to Assignment #2

Shifty business

Suppose $\mathbf{a} = [a_1, a_2, \dots, a_n]$ is an n-place row vector. The *left shift* of $\mathbf{a}$ by k places (where $0 \le k < n$) is the vector $\sigma_k(\mathbf{a}) = [a_{k+1}, a_{k+2}, \dots, a_n, a_1, a_2, \dots, a_k]$. For example, here is a left shift by 2 places of a 6-place vector:

$$\sigma_2([3, -1, 0, 5, 1, -2]) = [0, 5, 1, -2, 3, -1]$$

Note that for any vector $\mathbf{a}$, $\sigma_0(\mathbf{a}) = \mathbf{a}$.

1. Explain why $\sigma_k(s\mathbf{a}) = s\sigma_k(\mathbf{a})$ for any n-place vector $\mathbf{a}$, integer k with $0 \le k < n$, and scalar s. *[1.5]*

Solution. Suppose $\mathbf{a} = [a_1, a_2, \dots, a_n]$ is an n-place vector, s is a scalar, and k is an integer with $0 \le k < n$. Then

$$\begin{aligned}
\sigma_k(s\mathbf{a}) &= \sigma_k\left(s\,[a_1, a_2, \dots, a_k, a_{k+1}, \dots, a_n]\right) \\
&= \sigma_k\left([sa_1, sa_2, \dots, sa_k, sa_{k+1}, \dots, sa_n]\right) \\
&= [sa_{k+1}, \dots, sa_n, sa_1, sa_2, \dots, sa_k] \\
&= s\,[a_{k+1}, \dots, a_n, a_1, a_2, \dots, a_k] \\
&= s\sigma_k\left([a_1, a_2, \dots, a_k, a_{k+1}, \dots, a_n]\right) \\
&= s\sigma_k(\mathbf{a}),
\end{aligned}$$

as desired. ■

2. Explain why $\sigma_k(\mathbf{a} + \mathbf{b}) = \sigma_k(\mathbf{a}) + \sigma_k(\mathbf{b})$ for any n-place vectors $\mathbf{a}$ and $\mathbf{b}$, and any integer k with $0 \le k < n$. *[1.5]*

Solution. Suppose $\mathbf{a} = [a_1, a_2, \dots, a_n]$ and $\mathbf{b} = [b_1, b_2, \dots, b_n]$ are n-place vectors and k is an integer with $0 \le k < n$. Then

$$\begin{aligned}
\sigma_k(\mathbf{a} + \mathbf{b}) &= \sigma_k\left([a_1, a_2, \dots, a_k, a_{k+1}, \dots, a_n] + [b_1, b_2, \dots, b_k, b_{k+1}, \dots, b_n]\right) \\
&= \sigma_k\left([a_1 + b_1, a_2 + b_2, \dots, a_k + b_k, a_{k+1} + b_{k+1}, \dots, a_n + b_n]\right) \\
&= [a_{k+1} + b_{k+1}, \dots, a_n + b_n, a_1 + b_1, a_2 + b_2, \dots, a_k + b_k] \\
&= [a_{k+1}, \dots, a_n, a_1, a_2, \dots, a_k] + [b_{k+1}, \dots, b_n, b_1, b_2, \dots, b_k] \\
&= \sigma_k\left([a_1, a_2, \dots, a_k, a_{k+1}, \dots, a_n]\right) + \sigma_k\left([b_1, b_2, \dots, b_k, b_{k+1}, \dots, b_n]\right) \\
&= \sigma_k(\mathbf{a}) + \sigma_k(\mathbf{b}),
\end{aligned}$$

as desired. ■

3. Is it true that $\sigma_k(\sigma_\ell(\mathbf{a})) = \sigma_{k+\ell}(\mathbf{a})$? Explain why or why not. *[2]*

Solution. It is true. Suppose $\mathbf{a} = [a_1, a_2, \dots, a_n]$ is an n-place vector, and suppose that k and ℓ are integers such that each of k, ℓ, and $k+\ell$ is ≥ 0 and $< n$. Then:

$$\begin{aligned}
\sigma_k(\sigma_\ell(\mathbf{a})) &= \sigma_k(\sigma_\ell([a_1, a_2, \dots, a_\ell, a_{\ell+1}, \dots, a_{\ell+k}, a_{\ell+k+1}, \dots, a_n])) \\
&= \sigma_k([a_{\ell+1}, \dots, a_{\ell+k}, a_{\ell+k+1}, \dots, a_n, a_1, a_2, \dots, a_\ell]) \\
&= [a_{\ell+k+1}, \dots, a_n, a_1, a_2, \dots, a_\ell, a_{\ell+1}, \dots, a_{\ell+k}] \\
&= \sigma_{\ell+k}([a_1, a_2, \dots, a_\ell, a_{\ell+1}, \dots, a_{\ell+k}, a_{\ell+k+1}, \dots, a_n]) \\
&= \sigma_{\ell+k}(\mathbf{a}) \\
&= \sigma_{k+\ell}(\mathbf{a}) \qquad \text{(Since } k+\ell = \ell+k.\text{)} \ \blacksquare
\end{aligned}$$

4. How does the left shift operator interact with the dot product? *[2]*

Solution. The best one can hope for – and that turns out to actually work! –is probably that if $\mathbf{a} = [a_1, a_2, \dots, a_n]$ and $\mathbf{b} = [b_1, b_2, \dots, b_n]$ are n-place vectors and k is an integer with $0 \leq k < n$, then $\sigma_k(\mathbf{a}) \cdot \sigma_k(\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$:

$$\begin{aligned}
\sigma_k(\mathbf{a}) \cdot \sigma_k(\mathbf{b}) &= \sigma_k([a_1, a_2, \dots, a_k, a_{k+1}, \dots, a_n]) \cdot \sigma_k([b_1, b_2, \dots, b_k, b_{k+1}, \dots, b_n]) \\
&= [a_{k+1}, \dots, a_n, a_1, a_2, \dots, a_k] \cdot [b_{k+1}, \dots, b_n, b_1, b_2, \dots, b_k] \\
&= a_{k+1}b_{k+1} + a_{k+2}b_{k+2} + \cdots + a_nb_n + a_1b_1 + a_2b_2 + \cdots + a_kb_k \\
&= a_1b_1 + a_2b_2 + \cdots + a_kb_k + a_{k+1}b_{k+1} + a_{k+2}b_{k+2} + \cdots + a_nb_n \\
&= [a_1, a_2, \dots, a_k, a_{k+1}, \dots, a_n] \cdot [b_1, b_2, \dots, b_k, b_{k+1}, \dots, b_n] \\
&= \mathbf{a} \cdot \mathbf{b}
\end{aligned}$$

What about possibilities that turn out to not to work? For one, it isn't hard to come up with examples showing that if $k \neq \ell$, then $\sigma_k(\mathbf{a}) \cdot \sigma_\ell(\mathbf{b})$ need not equal $\mathbf{a} \cdot \mathbf{b}$. (For one example, try $\mathbf{a} = \mathbf{b} = [1, 2, 3, 4]$, $k = 1$, and $\ell = 2$.) $\blacksquare$

5. Find a vector $\mathbf{a}$ with as many places as you can such that each entry of $\mathbf{a}$ is either $+1$ or -1, and such that for every left shift by $k > 0$ places we have $|\mathbf{a} \cdot \sigma_k(\mathbf{a})| \leq 1$. *[3]*

Solution. Here's a table with essentially all the vectors with the given property of length n less than 39, where $+$ represents an entry of $+1$ and $-$ represents an entry of -1. ("Essentially all" means that every other vector of these lengths with the given property is obtained from one of those given below by reversing the order of the entries, and/or multiplying the vector by -1, and/or left- or right-shifting the vector.)

n	Vector
1	+
2	+−
3	+ + −
4	+ + +−
5	+ + + − +
7	+ + + − − + −
11	+ + + − − − + − − + −
13	+ + + + + − − + + − + − +
13	+ + + + + − + + + − − + −
15	+ + + + − + − + + − − + − − −
19	+ + + + − − + + − + + − − − − + − + −
23	+ + + + + − − − − + − + − − + + − − + + − + −
31	+ + + + − − − + − + − + + + − − − − + − − + − − + + + − + + −
31	+ + + + + − − − + + − + + + − + − + − − − − + − − + − + + − −
31	+ + + + + − − + − − + + − − − − + − + + − + − + − − − + + + −
31	+ + + + + − + + − − + + + − − − − + + − + − + − − + − − − + −
35	+ + + + + − − − + + + − + − − + − − + + − + − + − − − − + − − + + + −

The vectors of length ≤ 13 (except for the second one of length 13) on this list were discovered first, by R.H. Barker [**1**], in connection with work he was doing on radar. These actually satisfy a stronger property: that for every m with $0 < m < n$, the sum

$$a_1 a_{n-m+1} + a_2 a_{n-m+2} + \cdots + a_m a_n = \sum_{i=1}^{m} a_i a_{n-m+i}$$

has absolute value at most 1. It was later shown by R.J. Turyn and J. Storer [**2**] that these *Barker codes* (along with ++) are essentially the only vectors of +1s and −1s with this property.

The other vectors in the table, plus various existence results, were found after an engineer (my brother) working on a telecommunications protect wanted sequences with the property we were given, but could only find the finitely many Barker codes. There turn out to be *weak Barker codes* of arbitrarily large lengths.

References

1. R.H. Barker, *Synchronizing arrangements for pulse code systems*, US patent 2 721 318, issued Oct. 18, 1955 (filed Feb. 16, 1953).
2. R.J. Turyn and J. Storer, *On binary sequences*, Proc. Amer. Math. Soc. **12** (1961), pp. 394-399.
3. N. Bilaniuk, S. Bilaniuk, and B. Zhou, *Extensions to the known binary sequences of the Barker type* [This title better change!], unpublished.

That's all for now folks! ■