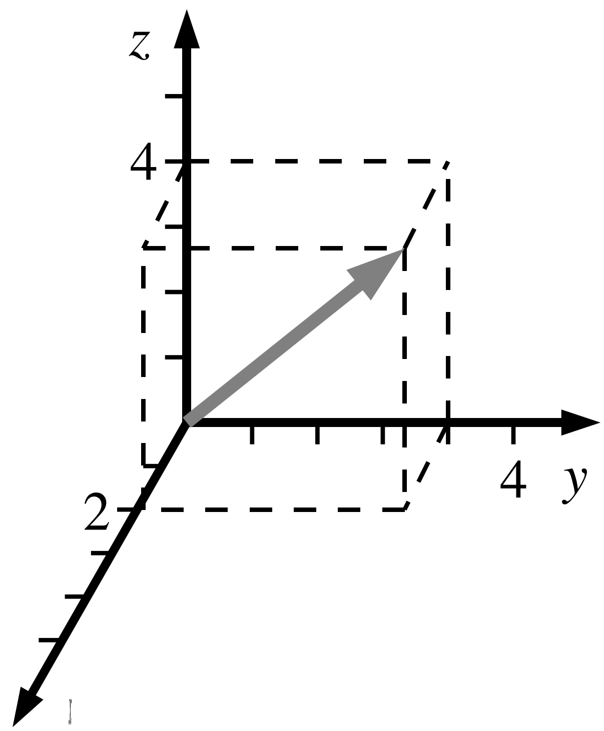
Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2008

Solutions to the quizzes

Quiz #1. Friday, 19 September, 2008. *[5 minutes]*

1. Sketch the vector $\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$ (in standard position) and find its length. *[5]*

Solution. Here's a sketch of the vector:



It remains to compute the length of the vector:

$$\left\| \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} \right\| = \sqrt{2^2 + 4^2 + 4^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6 \qquad \blacksquare$$

Quiz #2. Friday, 26 September, 2008. *[5 minutes]*

1. Let $\mathbf{a} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$. Compute $\text{proj}_{\mathbf{b}}(\mathbf{a})$. *[5]*

Solution. We plug into the formula and chug away!

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} = \frac{\begin{bmatrix} 3 \\ -1 \\ 0 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$= \frac{3 \cdot 1 + (-1) \cdot 0 + 0 \cdot 1 + (-3) \cdot (-1)}{1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 + (-1) \cdot (-1)} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$= \frac{6}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -2 \end{bmatrix}$$

Whew! ■

Quiz #3. Friday, 3 October, 2008. *[5 minutes]*

1. Find the (least) distance from the point $(0,0,0)$ to the plane $x+y+z=12$. *[5]*

Solution. First, observe that $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a vector perpendicular to the plane $x+y+z = 12$: just read off the coefficients of the variables in the given equation.

Second, it is easy to see that $(12,0,0)$ is a point on the plane, and hence that $\mathbf{a} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$ is the vector that runs from the given point, $(0,0,0)$, to the point $(12,0,0)$ on the plane.

Third, to obtain the vector that runs between $(0,0,0)$ and the nearest point to it on the plane (in one direction or the other), we project the vector $\mathbf{a}$ onto the normal vector $\mathbf{b}$:

$$\text{proj}_{\mathbf{n}}(\mathbf{a}) = \frac{\mathbf{a}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} = \frac{\begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{12}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

Finally, the distance between $(0,0,0)$ and the plane $x+y+z=12$ is the length of this projection vector:

$$\left\|\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}\right\| = \sqrt{4^2+4^2+4^2} = 4\sqrt{3} \qquad \blacksquare$$

Quiz #4. Friday, 10 October, 2008. *[5 minutes]*

1. Solve the following system of linear equations using Gauss-Jordan reduction. *[5]*

$$\begin{array}{ccccccccc} x & & & + & z & & & = & 0 \\ & & y & & & - & w & = & 0 \\ x & & & - & z & & & = & 1 \\ & & y & & & + & w & = & 1 \end{array}$$

Solution. We set up the corresponding augmented matrix and throw elementary row operations at it ...

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_3 - R_1 \\ R_4 - R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ -\frac{1}{2}R_3 \\ \frac{1}{2}R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

$$\begin{array}{c} R_1 - R_3 \\ R_2 + R_4 \\ \Longrightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

Hence the (only!) solution to the given system of linear equations is $x = y = w = \frac{1}{2}$ and $z = -\frac{1}{2}$. ■

Quiz #5. Friday, 17 October, 2008. *[10 minutes]*

1. Determine whether $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \right\}$ or not. *[5]*

Solution. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \right\}$ exactly when there are scalars a, b, and c such that

$$a \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} .$$

We set up the corresponding system of equations in augmented matrix form and attempt to solve it:

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 4 & 1 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 0 & 4 & 1 \end{array}\right]$$

$$\underset{R_3 - 2R_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -4 & 4 & -1 \end{array}\right] \overset{R_1 - 2R_2}{\underset{R_3 + 4R_2}{\Longrightarrow}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

At this point the Gauss-Jordan algorithm can take us no further. Observe that the last row corresponds to the equation $0a + 0b + 0c = 3$, which is impossible to satisfy, so there are no solutions. It follows that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is *not* in Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \right\}$. ∎

Quiz #6. Friday, 31 October, 2008. *[10 minutes]*

1. Use the Gauss-Jordan method to find the inverse of $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. *[5]*

Solution. We set up the super-augmented matrix and go:

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_3 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 5 & -5 & 0 & 1 & -2 \\ 0 & 4 & -4 & 1 & 0 & -3 \end{array}\right] \begin{array}{c} \Longrightarrow \\ \frac{1}{5} \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & \frac{1}{5} & -\frac{2}{5} \\ 0 & 4 & -4 & 1 & 0 & -3 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_3 - 4R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & \frac{1}{5} & -\frac{2}{5} \\ 0 & 0 & 0 & 1 & -\frac{4}{5} & -\frac{7}{5} \end{array}\right]$$

At this point it should be apparent that we cannot turn the left-hand matrix into the identity matrix. It follows that $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ has no inverse. ■

Quiz #7. Friday, 7 November, 2008. *[10 minutes]*

1. Suppose $\mathbf{A}$ is an $n \times n$ matrix which has an inverse, and suppose $\mathbf{a}_1$, $\mathbf{a}_2$, ... , $\mathbf{a}_n$ are the columns of $\mathbf{A}$. What can you deduce about this collection of vectors? *[5]*

Solution. Recall from class (and the text) that there is a longish list of properties which are equivalent to a matrix having an inverse. A couple of these are relevant here:

$$\begin{aligned} \mathbf{A} \text{ has an inverse} &\iff \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \text{ are linearly independent} \\ &\iff \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \mathbb{R}^n \end{aligned}$$

One can rephrase these a little, using the definitions of independence and span:

$$\iff \text{No column of } \mathbf{A} \text{ is a linear combination of the other columns of } \mathbf{A}.$$
$$\iff \text{Every vector } \mathbf{b} \in \mathbb{R}^n \text{ is a linear combination of the columns of } \mathbf{A}.$$

One could also note another equivalent that is closely related:

$$\iff \text{The dimension of Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \text{ is } n.$$

Using the fact that the (transposes of the) columns of $\mathbf{A}$ are the rows of $\mathbf{A}^T$, we can also make a similar statements about the rows of $\mathbf{A}^T$. ■

Quiz #8. Friday, 8 November, 2008. *[5 minutes]*

1. Determine whether $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; |x| = |y| \right\}$ is a subspace of $\mathbb{R}^2$ or not. *[5]*

Solution. U is not a subspace of $\mathbb{R}^2$. While it is closed under multiplication by scalars, it is not closed under vector addition. For example, it is easy to see that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are in U, since $|1| = 1 = |1|$ and $|-1| = 1 = |1|$. However,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

which is not in U because $|0| = 0 \neq 2 = |2|$. ■

Quiz #9. Friday, 21 November, 2008. *[10 minutes]*

1. Find a spanning set for the subspace

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; \begin{array}{rcrcrcl} & x & + & 2y & & & = & 0 \\ & & & 2y & + & z & = & 0 \\ - & x & + & 2y & + & 2z & = & 0 \end{array} \right\}$$

of $\mathbb{R}^3$. *[5]*

Solution. We find the (parametric representation of the) set of solutions of the system of linear equations used to define S:

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -1 & 2 & 2 & 0 \end{array}\right] \underset{R_3 + R_1}{\Longrightarrow} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ \Longrightarrow \\ R_3 - 2R_2 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\frac{1}{2}R_2}{\Longrightarrow} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last augmented matrix corresponds to equations $x - z = 0$ and $y + \frac{1}{2}z = 0$, which we can use to solve for x and y in terms of z: $x = z$ and $y = -\frac{1}{2}z$. hence the vector-parametric representation of the set of solutions to the original system of equations is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix},$$

where $t \in \mathbb{R}$ is the paramenter. It follows that $S = \text{Span}\left\{ \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$. ■

Quiz #10. Friday, 28 November, 2008. *[12 minutes]*

1. Determine the dimension of the subspace

$$S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

of $\mathbb{R}^4$. *[5]*

Solution. There are several ways to do this. In this solution we will find a basis for S that is a subset of the given spanning set for S and then count how many vectors there are in this basis. To this end we assemble the given spanning set into the columns of a matrix and then row reduce this matrix as far as possible:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \underset{R_3 - R_1}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{matrix} R_1 - R_2 \\ \Longrightarrow \\ R_3 + R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \underset{(-1)R_3}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{matrix} R_1 - R_3 \\ \Longrightarrow \\ \\ R_4 + R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that there are three non-zero rows remaining in this matrix. These have leading non-zero entries in the first three columns; thus the first three vectors of the given spanning set,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\},$$

will serve as a basis for S. It follows that the dimension of S is three. ■

Quiz #11. Friday, 5 December, 2008. *[10 minutes]*

1. Find the the eigenvalue(s) and all the eigenvectors of $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$. *[5]*

Solution. First, we find the eigenvalues.

$$\left| \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = (2-\lambda)(-\lambda) - (-1) \cdot 1 = \lambda^2 - 2\lambda + 1$$

It's pretty easy to see that $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$ only when $\lambda = 1$, and so $\lambda = 1$ is the only eigenvalue of the given matrix.

Second, we find all the eigenvectors of the given matrix for the eigenvalue $\lambda = 1$; that is, we find all the solutions $\begin{bmatrix} x \\ y \end{bmatrix}$ of the equation

$$\left(\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \textit{i.e.} \quad \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It's obvious that we are looking for all the solutions to the linear equation $x - y = 0$, *i.e.* $x = y$. If we let y equal the parameter t, it follows that $x = t$ as well, and so the vector-parametric presentation of the set of solutions is $\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where $t \in \mathbb{R}$. ■