Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2008

FINAL EXAMINATION
Friday, 19 December, 2008

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Show all your work. *If in doubt about something,* **ask!**

Aids: Calculator; annotated *Formula for Success* or $8.5'' \times 11''$ aid sheet; one brain.

Part I. Do all of **1**–**5**.

1. Consider the plane in $\mathbb{R}^3$ given by the equation $x + y + 2z = 8$, and the line in $\mathbb{R}^3$ given by the parametric equations $x = 4 + t$, $y = 4 - t$, and $z = 0$.

a. Sketch the parts of this plane and line that lie in the first octant. *[4]*

b. Find the angle between the normal vector of the plane and the direction vector of the line. *[3]*

c. Show that the line is contained in the plane. *[3]*

2. Consider the following system of linear equations.

$$\begin{array}{rcrcrcrcr} w & + & & & 2y & + & 4z & = & -8 \\ & & x & - & 3y & - & z & = & 6 \\ 3w & + & 4x & - & 6y & + & 8z & = & 0 \\ & - & x & + & 3y & + & 4z & = & -12 \end{array}$$

Use Gauss-Jordan elimination to find all the solutions, if any, of this system. Without computing it, what must the determinant of the matrix of coefficients be? *[10]*

3. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$.

a. Find the inverse of **A**, if it exists. *[10]*

b. Use your work in **a** to compute the determinant of **A**. *[5]*

4. Find all the eigenvalues and eigenvectors of $\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$. *[15]*

5. Let $\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

a. Find a basis for the column space of **M**. *[10]*

b. Determine the rank and nullity of **M**. *[5]*

Part II. Do any *three* of **6–11**.

6. Suppose T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad \text{and}\quad T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Find the matrix $\mathbf{D}$ such that for all $\mathbf{x} \in \mathbb{R}^3$, $T(\mathbf{x}) = \mathbf{D}\mathbf{x}$. *[10]*

7. Suppose $\mathbf{B}$ is an invertible matrix. Show that $\left(\mathbf{B}^T\right)^{-1} = \left(\mathbf{B}^{-1}\right)^T$. *[10]*

8. Sketch the point $(0, 5, 3)$ and the line given by the parametric equations $x = 2t$, $y = -6t$, and $z = 10t$, where $t \in \mathbb{R}$, and find the distance between them. *[10]*

9. Find a linearly independent subset of $\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}$ that is as large as possible. *[10]*

10. Give examples, if such exist, of invertible 3×3 matrices $\mathbf{P}$ and $\mathbf{Q}$ such that:

- **a.** $\mathbf{P} + \mathbf{Q}$ is invertible. *[2]*
- **b.** $\mathbf{PQ}$ is invertible. *[2]*
- **c.** $\mathbf{P} + \mathbf{Q}$ is not invertible. *[2]*
- **d.** $\mathbf{PQ}$ is not invertible. *[2]*
- **e.** $\mathbf{P}^2 + \mathbf{Q}^T$ is not invertible. *[2]*

11. Suppose $T : \mathbb{R}^n \to \mathbb{R}^k$ is a linear transformation. Explain why the null space of T, $\text{null}(T) = \{\, \mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0} \,\}$, is a subspace of $\mathbb{R}^n$. *[10]*

[Total = 95]

Part Max. Bonus!

Min. Write an original little poem about linear algebra or mathematics in general. *[2]*

HAVE A NICE BREAK!
I HOPE TO SEE YOU NEXT TERM!