Mathematics 135H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2007

Solutions to Assignment #3

Linear constraints and optimization

Recall that this assignment dealt with the solid whose faces are (parts of) the planes given by the equations $x = 0$, $y = 0$, $z = 0$, $x + y = 10$, $x + z = 10$, $y + z = 10$, and $x + y + z = 14$. It consists of the set of points with coordinates (x, y, z) satisfying all of the seven inequalities $x \geq 0$, $y \geq 0$, $z \geq 0$, $x + y \leq 10$, $x + z \leq 10$, $y + z \leq 10$, and $x + y + z \leq 14$.

1. Find the coordinates of all of the vertices of this solid and make as accurate a sketch as you can of it. *[6]*

Solution. The vertices of the solid are those points where at least three of the given planes intersect, and whose coordinates satisfy all of the given inequalities. We therefore need to consider possible intersections of three (or more) of the given planes at a time. With seven planes there are 35 different ways to choose three at a time, but the symmetries among the variables will let us cut down the amount of work a fair bit. We'll name the points that make the cut as we go along.

i. $x = 0$, $y = 0$, and $z = 0$ obviously intersect in the point $O = (0, 0, 0)$ and the coordinates of this point satisfy all the given inequalities.

ii. $x = 0$, $y = 0$, and $x + y = 10$ obviously can't intersect at the same point, since otherwise we'd have $0 + 0 = 10$. Similarly, $x = 0$, $z = 0$, and $x + z = 10$ can't intersect at the same point, and neither can $y = 0$, $z = 0$, and $y + z = 10$.

iii. $x = 0$, $y = 0$, $x + z = 10$, and $y + z = 10$ all intersect in the point $C = (0, 0, 10)$, and the coordinates of this point satisfy all the given inequalities. Similarly, the point of intersection of $x = 0$, $z = 0$, $x + y = 10$, and $y + z = 10$, $B = (0, 10, 0)$, satisfies all the given inequalities, and so does the point of intersection of $y = 0$, $z = 0$, $x + z = 10$, and $y + z = 10$, $A = (10, 0, 0)$.

iv. $x = 0$, $x + y = 10$, and $x + z = 10$ all intersect in the point $(0, 10, 10)$, but the coordinates of this point do not satisfy the inequality $y + z \leq 10$, so this point is not in the solid we're investigating. Similarly, the point of intersection of $y = 0$, $x + y = 10$, and $y + z = 10$, $(10, 0, 10)$, is not in the solid, and neither is the point of intersection of $z = 0$, $x + z = 10$, and $y + z = 10$, $(10, 10, 0)$.

v. $x = 0$, $y = 0$, and $x + y + z = 14$ all intersect in the point $(0, 0, 14)$, but the coordinates of this point do not satisfy the inequality $x + z \leq 10$ (nor $y + z \leq 10$ either), so this point is not in the solid. Similarly, the point of intersection of $x = 0$, $z = 0$, and $x + y + z = 14$, $(0, 14, 0)$, is not in the solid, and neither is the point of intersection of $y = 0$, $z = 0$, and $x + y + z = 14$, $(14, 0, 0)$.

vi. $x = 0$, $x + y = 10$, and $x + y + z = 14$ intersect in the point $(0, 10, 4)$, but the coordinates of this point do not satisfy the inequality $y + z \leq 10$, so this point is not in the solid. Similarly, the point of intersection of $x = 0$, $x + z = 10$, and $x + y + z = 14$, $(0, 4, 10)$, is not in the solid. Neither is the point of intersection of $y = 0$, $x + y = 10$, and $x + y + z = 14$, $(10, 0, 4)$, nor the point of intersection of $y = 0$, $y + z = 10$, and $x + y + z = 14$, $(4, 0, 10)$, nor the point of intersection of $z = 0$, $x + z = 10$, and $x + y + z = 14$, $(10, 4, 0)$, nor the point of intersection of $z = 0$, $y + z = 10$, and $x + y + z = 14$, $(4, 10, 0)$.

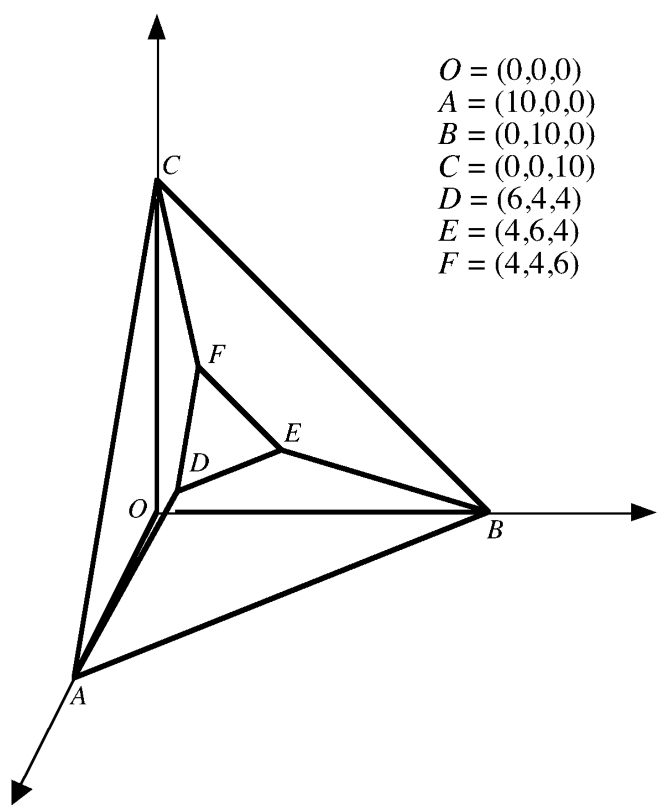
vii. $x = 0$, $y + z = 10$, and $x + y + z = 14$ obviously can't intersect in the same point, since otherwise $0 + 10 = 14$. Similarly, $y = 0$, $x + z = 10$, and $x + y + z = 14$ can't intersect in the same point, and neither can $z = 0$, $x + y = 10$, and $x + y + z = 14$.

viii. $x + y = 10$, $y + z = 10$, and $x + y + z = 14$ intersect in the point $E = (4, 6, 4)$, and the coordinates of this point satisfy all the given inequalities. Similarly, the point of intersection of $x + y = 10$, $x + z = 10$, and $x + y + z = 14$, $D = (6, 4, 4)$, satisfies all the given inequalities, and so does the point of intersection of $x + z = 10$, $y + z = 10$, and $x + y + z = 14$, $F = (4, 4, 6)$.

ix. Finally, $x + y = 10$, $x + z = 10$, and $y + z = 10$ intersect in the point $E = (5, 5, 5)$, but the coordinates of this point do not satisfy the inequality $x + y + z \leq 14$.

Note that items *i*–*ix* between them use up $1 + 3 + 12 + 3 + 3 + 6 + 3 + 3 + 1 = 35$ different ways of taking three of the seven given planes at a time, so we have considered all the possibilities.

In the end, we have seven points that matter; graphing them with the axes in standard position and connecting up the dots (except for those connections passing inside the solid) gives a picture like the following:



This solid is a *heptahedron*, that is, it has seven faces: the trianges OAB, OAC, OBC, and DEF and the quadrangles $ABED$, $ACFD$, and $BCFE$. ■

2. Find the maximum value of the function $f(x, y, z) = 2x + 2y + 3z$ on this solid and determine at which point(s) of the solid this maximum occurs. *[4]*

Solution. It's obvious that $f(0, 0, 0) = 0$ is not the maximum value of $f(x, y, z)$ on the solid. We'll find a candidate for the maximum by moving from vertex to vertex of the solid, starting with $O = (0, 0, 0)$, trying to increase the value of $f(x, y, z)$ as much as possible at each step.

The vertices neighbouring $O = (0, 0, 0)$ are $A = (10, 0, 0)$, $B = (0, 10, 0)$, and $C = (0, 0, 10)$. The values of $f(x, y, z)$ at these points are 20, 20, and 30, respectively, all of which are larger than 0, the value at $O = (0, 0, 0)$. Since the largest of these is 30, we move to $C = (0, 0, 10)$ and repeat the process.

The vertices neighbouring $C = (0, 0, 10)$ are $O = (0, 0, 0)$, $A = (10, 0, 0)$, $B = (0, 10, 0)$, and $F = (4, 4, 6)$. The values of $f(x, y, z)$ at these points are 0, 20, 20, and 34, respectively. Since the only one of these larger than 30 – the value at $C = (0, 0, 10)$ – of these is 34, we move to $F = (4, 4, 6)$ and repeat the process.

The vertices neighbouring $F = (4, 4, 6)$ are $C = (0, 0, 10)$, $D = (6, 4, 4)$, and $E = (4, 6, 4)$. The values of $f(x, y, z)$ at these points are 30, 32, and 32, respectively. Since none of these is larger than 34 – the value at $F = (4, 4, 6)$ – it is time to stop: our candidate for the maximum of $f(x, y, z)$ on the solid is 34, achieved at the vertex $F = (4, 4, 6)$.

Is 34 the best we can get on the solid? Yes. Since every point on the solid satisfies the inequalities $x + z \leq 10$, $y + z \leq 10$, and $x + y + z \leq 14$,

$$\begin{aligned} f(x, y, z) = 2x + 2y + 3z &= (x + z) + (y + z) + (x + y + z) \\ &\leq 10 + 10 + 14 = 34\,. \end{aligned}$$

This means that the value of 34 achieved at the vertex $(4, 4, 6)$ is the largest possible value of $f(x, y, z)$ on the solid.

Moreover, at every point on the solid except for $(4, 4, 6)$, at least one of the the inequalities $x + z \leq 10$, $y + z \leq 10$, and $x + y + z \leq 14$ is strict, so the value of $f(x, y, z)$ at every other point is strictly less than 34. Hence $f(x, y, z)$ achieves the value of 34 on the solid *only* at the vertex $(4, 4, 6)$. ■

Note: It is a general fact that the maximum of a linear function on a solid defined by linear constraints occurs at a vertex. (It might also occur along an edge or a face of the solid, but these do include vertices.) If you want to learn more about this, and the "moving from vertex to vertex" procedure used above, which can be reduced to doing a lot of matrix manipulation, take *Mathematics-Science 335H: Linear programming*, whose only prerequisite is MATH 135H. Alternatively, you can read up on your own about "linear programming" and the "Simplex Method."