Mathematics 135H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2007

Solutions to Assignment #2

1. Using algebra – including linear algebra! – find the radius and centre of the circle passing through the points $(2,5)$, $(0,1)$, and $(3,4)$.

Solution. The first thing to do is to plug the (x,y)-coordinates of our three points into the equation for a circle – recall that the equation of a circle of radius r and with centre at (a,b) is $(x-a)^2+(y-b)^2=r^2$ – and expand:

$$\begin{aligned} r^2 &= (2-a)^2+(5-b)^2 = a^2+b^2-4a-10b+29 \\ r^2 &= (0-a)^2+(1-b)^2 = a^2+b^2-2b+1 \\ r^2 &= (3-a)^2+(4-b)^2 = a^2+b^2-6a-8b+25 \end{aligned}$$

Since the left-hand sides of these equations are equal, so are the right-hand sides. It follows that:

$$\begin{aligned} a^2+b^2-4a-10b+29 &= a^2+b^2-2b+1 \\ a^2+b^2-4a-10b+29 &= a^2+b^2-6a-8b+25 \\ a^2+b^2-6a-8b+25 &= a^2+b^2-2b+1 \end{aligned}$$

Cancelling lets us get rid of the a^2+b^2 throughout, and then rearranging leaves us with:

$$\begin{aligned} 4a+8b &= 28 \\ -2a+2b &= 4 \\ 6a+6b &= 24 \end{aligned}$$

Solving this system of linear equations for a and b (which we'll leave as an exercise for the reader), gives $a=1$ and $b=3$. We can plug this, along with the (x,y)-coordinates of one of our three points, into the equation of a circle to compute r^2:

$$r^2 = (0-1)^2+(1-3)^2 = 1^2+(-2)^2 = 1+4 = 5$$

Thus, the circle passing through the given points has centre $(1,3)$ and radius $\sqrt{5}$. ■

2. Explain why, in general, three points in two-dimensional space (not all in a straight line) suffice to specify a circle passing through all three.

Solution. Given any three different points in two-dimensional space, one can try to execute the procedure used in the solution to problem **1**. It ought to work (and does!) so long as the points aren't positioned in a way that makes it impossible for them all to be on a circle; namely, so long as they are not all on the same line.

Since circles (of finite radius!) are not straight, it obviously impossible for three different points on a straight line to be on the same circle.

On the other hand, suppose that we have three points, say (p,q), (s,t), and (u,v), which are *not* on the same line. If we follow the same procedure used in the solution to problem **1** above, we get down to a system of equations equivalent to:

$$\begin{aligned} 2(p-s)a+2(q-t)b &= (p^2+q^2)-(s^2+t^2) \\ 2(s-u)a+2(t-v)b &= (s^2+t^2)-(u^2+v^2) \\ 2(u-p)a+2(v-q)b &= (u^2+v^2)-(p^2+q^2) \end{aligned}$$

It's pretty easy to see that any two of these equations give you the third (just add the two you picked and multiply by -1), so we only need the first two. The matrix representation of the system given by the first two equations is:

$$\begin{bmatrix} 2(p-s) & 2(q-t) \\ 2(s-u) & 2(t-v) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (p^2+q^2)-(s^2+t^2) \\ (s^2+t^2)-(u^2+v^2) \end{bmatrix}$$

This system has an unique solution if the matrix $\begin{bmatrix} 2(p-s) & 2(q-t) \\ 2(s-u) & 2(t-v) \end{bmatrix}$ is non-singular; that is, if the column vectors $\begin{bmatrix} 2(p-s) \\ 2(s-u) \end{bmatrix}$ and $\begin{bmatrix} 2(q-t) \\ 2(t-v) \end{bmatrix}$ are linearly independent. This boils down to checking whether one of the two is a multiple of the other.

Suppose, for the sake of argument, that there was a number m such that

$$\begin{bmatrix} 2(p-s) \\ 2(s-u) \end{bmatrix} = m \begin{bmatrix} 2(q-t) \\ 2(t-v) \end{bmatrix} .$$

Writing this out coordinate by coordinate and cancelling those pesky 2s gives us:

$$p - s = m(q-t) \quad \text{and} \quad s - u = m(t-v)$$

Geometrically, this means that the line joining (p,q) and (s,t) has the same slope as the line joining (s,t) and (u,v); since these lines have the same slope and both pass through (s,t), they are the same line and all three points are on it. *Oops!* That cannot be: we are considering the case where the three points are *not* on the same line. So $\begin{bmatrix} 2(p-s) \\ 2(s-u) \end{bmatrix}$ and $\begin{bmatrix} 2(q-t) \\ 2(t-v) \end{bmatrix}$ can't be multiples of one another, which means they must be linearly independent.

Since $\begin{bmatrix} 2(p-s) \\ 2(s-u) \end{bmatrix}$ and $\begin{bmatrix} 2(q-t) \\ 2(t-v) \end{bmatrix}$ are linearly independent, $\begin{bmatrix} 2(p-s) & 2(q-t) \\ 2(s-u) & 2(t-v) \end{bmatrix}$ is non-singular, which means that the associated system of linear equations has an unique solution for a and b. This gives us the centre of the circle, and we can find r as in the solution to problem **1**.

You can try to figure out what happens if the three points aren't all different, since this solution is already serious overkill ... (A *lot* less than what is given above would have sufficed for full credit.) ■

3. How many points does one need, in general, to specify a sphere in three-dimensional space? What restriction(s) must these points satisfy?

Solution. In three-dimensional space, one can also use the same kind of procedure used in the solution to problem **1**. Three different points are not enough, though: think about resting balls of different sizes on the tops of three pillars spaced in an equilateral triangle.

Four points will do, though, so long as the points aren't positioned in a way that makes it impossible for them all to be on a sphere; namely, so long as they are not all on the same plane. ■

Bonus. Show that there are irrational numbers a and b such that a^b is rational.

Solution. (The cleverest one I know of ...) It is fairly easy to check that $\sqrt{2}$ is irrational. (If you haven't seen the argument in tutorial, ask about it in in office hours or look it up.) Consider $\sqrt{2}^{\sqrt{2}}$. If this number is rational, we are done because then $a = b = \sqrt{2}$ does the job. On the other hand, if it is not rational (*i.e.* irrational), then $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ do the job because:

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^{2} = 2$$

So, whether $\sqrt{2}^{\sqrt{2}}$ turns out to be rational or not, we can find irrational numbers a and b such that a^b is rational. ■