Mathematics 1121H – Calculus II

Trent University, Winter 2026

Solutions to Assignment #8

~~Series~~ Sequence Business ~~IV~~

Due on Friday, 13 March.

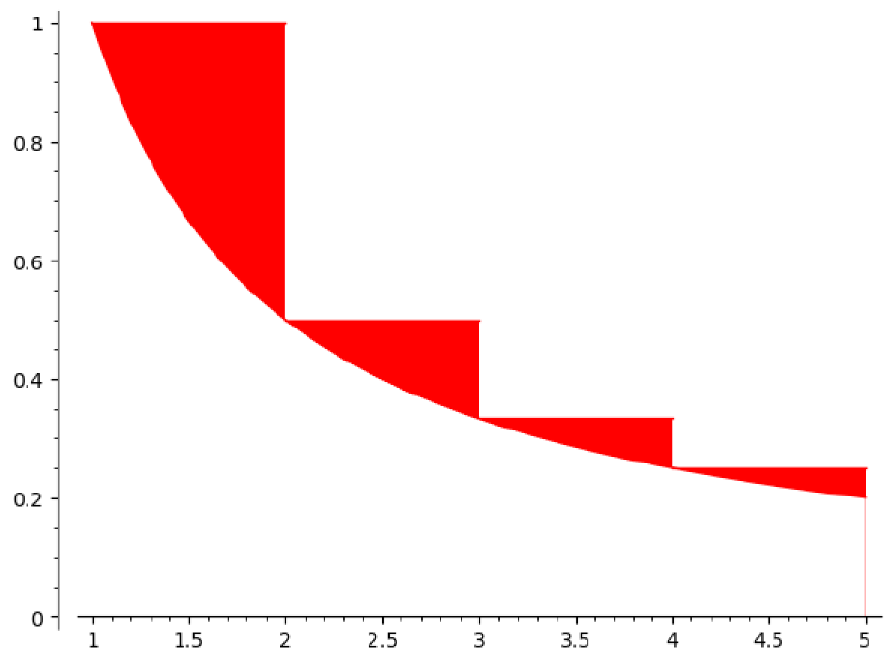
Recall from Assignment #6 that the sequence of partial sums of the harmonic series, $H_n = \sum_{k=1}^{n} \frac{1}{k}$, diverges, *i.e.* $\lim_{n\to\infty} H_n = \infty$. This assignment is about the related sequence $\gamma_n = H_{n-1} - \ln(n)$, which converges.

1. Write a SageMath subroutine that plots γ_n as an area for $n \geq 2$. Test it for $n = 5$, 10, and 20. *[4]*

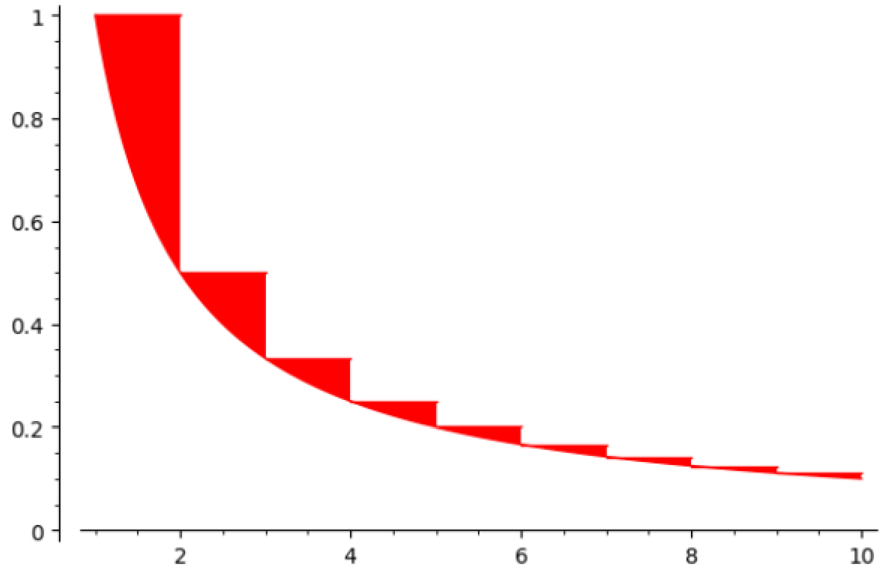
Hint. $\ln(n) = \int_1^n \frac{1}{x}\,dx$.

Solution. γ_n is the area of the left-hand rule Riemann rectangles for $\frac{1}{x}$ for $1 \leq x \leq n$ minus, as $\ln(n) = \int_1^n \frac{1}{x}\,dx$, the area under $\frac{1}{x}$ proper.

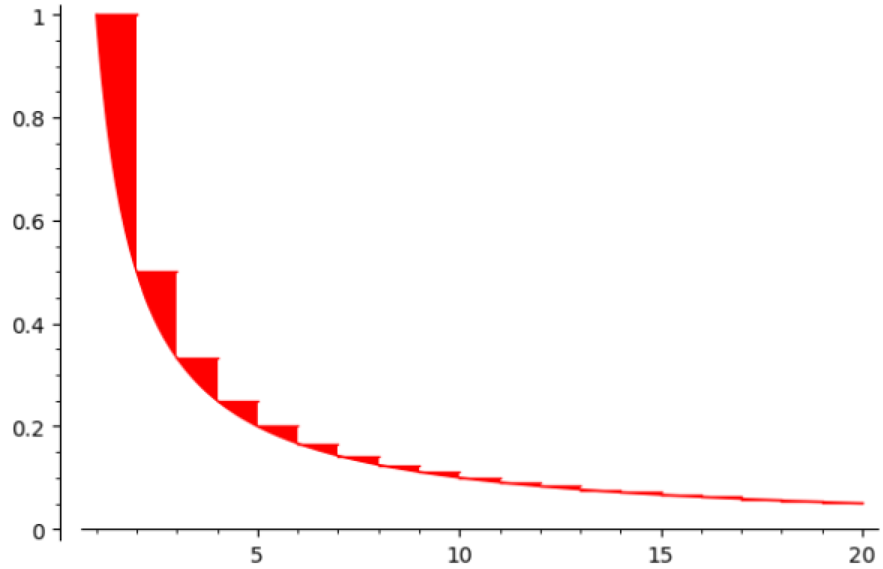
```
[1]: var('k,n,x')
def gamma_plotter( n ):
    k = 1
    p = plot( 1/k, k, k+1, color='red', fill=True, fillcolor='red', fillalpha=1␣
 ↪)
    while ( k < n ):
        p = p + plot( 1/k, k, k+1, color='red', fill=True, fillcolor='red', ␣
 ↪fillalpha=1 )
        k = k + 1
    p = p + plot( 1/x, 1, n, color='red', fill=True, fillcolor='white',␣
 ↪fillalpha=1 )
    show( p )
gamma_plotter(5)
```



```
[2]: gamma_plotter(10)
```




```
[3]: gamma_plotter(20)
```



□

2. Explain why $\gamma_n = \int_1^n \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x} \right) dx$ for $n \geq 2$, where $\lfloor x \rfloor = \text{floor}(x)$ is the greatest integer $\leq x$. *[2]*

HINT. What is $\lfloor x \rfloor$ for $k \leq x < k+1$?

Solution. When $k \le x < k+1$, $\lfloor x \rfloor = k$, so $\dfrac{1}{\lfloor x \rfloor} = \dfrac{1}{k}$. It follows that

$$\begin{aligned}\int_1^n \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x}\right) dx &= \int_1^n \frac{1}{\lfloor x \rfloor}\, dx - \int_1^n \frac{1}{x}\, dx \\ &= \int_1^2 \frac{1}{1}\, dx + \int_2^3 \frac{1}{2}\, dx + \cdots + \int_{n-1}^n \frac{1}{n-1}\, dx - \ln(n) \\ &= \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1} - \ln(n) = H_{n-1} - \ln(n) = \gamma_n \quad \square\end{aligned}$$

3. Show – by hand! – that $\gamma = \lim_{n\to\infty} \gamma_n$ exists, and that $0.5 < \gamma < 1$. *[4]*

Hint. To show that $0.5 < \gamma$, it is good enough to find some n for which $0.5 < \gamma_n$. (Why?) For the other side, show that $\gamma < 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \cdots$. The Monotone Convergence Theorem may also come in handy.

Solution. Since $\gamma_n < \gamma_{n+1} < \gamma$ for all $n \ge 2$ – we add a little more area to γ_n to get γ_{n+1} – it suffices to find an n such that $0.5 < \gamma_n$ to get $0.5 < \gamma$. $n = 8$ will do the job:

```
[4]: clear_vars()
     var('k')
     N( sum( 1/k, k, 1, 7 ) - log(8) )
```

```
[4]: 0.513415601177307
```

It remains to show that $\gamma < 1$. Note that when $k \le x \le k+1$, $\dfrac{1}{k+1} \le \dfrac{1}{x} \le \dfrac{1}{k}$. With the help of question **2**, it follows that

$$\begin{aligned}\gamma = \lim_{n\to\infty} \gamma_n &= \lim_{n\to\infty} \int_1^n \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x}\right) dx = \lim_{n\to\infty} \sum_{k=1}^{n-1} \int_k^{k+1} \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x}\right) dx \\ &< \lim_{n\to\infty} \sum_{k=1}^{n-1} \int_k^{k+1} \left(\frac{1}{k} - \frac{1}{k+1}\right) dx = \lim_{n\to\infty} \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \lim_{n\to\infty} \left[\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{(n-1)+1}\right)\right] \\ &= \lim_{n\to\infty} \left[1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n}\right] = \lim_{n\to\infty} \left[1 - \frac{1}{n}\right] = 1 - 0 = 1,\end{aligned}$$

as desired. The above, and the fact that $\gamma_n < \gamma_{n+1}$ for all $n \ge 2$, means the limit exists by the Monotone Convergence Theorem since 1 is an upper bound for all the γ_n. $\square$

Bonus. Verify by hand that $\gamma = \lim_{n\to\infty} \gamma_n = -\int_0^\infty e^{-x} \ln(x)\, dx$, showing all the essential steps. *[2]*

No Solution. This bonus is open until the end of classes this term. ■