Mathematics 1121H – Calculus II
TRENT UNIVERSITY, Winter 2026

Solutions to Assignment #7
Series Business III
Due on Friday, 6 March.

This assignment is concerned with the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, where $n! = n(n-1)(n-2)\cdots 2\cdot 1$ for positive integers n and $0! = 1$, mainly to make formulas like the one for series work without having to write exceptions when $n = 0$.

1. What is the sum of this series? *[1]*

HINT. Ask SageMath!

SOLUTION. SageMath says the sum of the series is e^x:

```
[1]: var('x')
     var('n')
     sum( x^n/factorial(n), n, 0, oo )

[1]: e^x
```

□

2. Prove that the given series is equal to the function obtained in answering question **1** by showing that they both satisfy the differential equation $\frac{dy}{dx} = y$ with initial condition $y = 1$ when $x = 0$. *[3]*

NOTE. We're exploiting the fact that there is only one solution to a given differential equation with given initial conditions. Proving that is a bit beyond the scope of this course ...

SOLUTION. First, $\frac{d}{dx}e^x = e^x$ and $e^0 = 1$, so $y = e^x$ satisfies the given differential equation and initial condition.

Second, note that $(n+1)! = (n+1)\cdot n!$ for all $n \geq 0$ and recall the conventions that $0! = 1$ and $0^0 = 1$. It follows that

$$\begin{aligned}
\frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{x^n}{n!}\right) \\
&= \frac{d}{dx}\left(\frac{x^0}{0!}\right) + \frac{d}{dx}\left(\frac{x^1}{1\cdot 0!}\right) + \frac{d}{dx}\left(\frac{x^2}{2\cdot 1!}\right) + \frac{d}{dx}\left(\frac{x^3}{3\cdot 2!}\right) + \frac{d}{dx}\left(\frac{x^4}{4\cdot 3!}\right) + \cdots \\
&= \frac{d}{dx}1 + \frac{1x^0}{1\cdot 0!} + \frac{2x^1}{2\cdot 1!} + \frac{3x^2}{3\cdot 2!} + \frac{4x^3}{4\cdot 3!} + \cdots \\
&= 0 + \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!},
\end{aligned}$$

so the sum satisfies the given differential equation. Since

$$\sum_{n=0}^{\infty} \frac{0^n}{n!} = \frac{0^0}{0!} + \frac{0^1}{1!} + \frac{0^2}{2!} + \frac{0^3}{3!} + \cdots = \frac{1}{1} + 0 + 0 + 0 + \cdots = 1,$$

the sum also satisfies the given initial condition.

Since they both satisfy the same differential equation and initial condition, these two functions of x must be equal, so $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. □

3. What is a_n in terms of n if $\sum_{n=0}^{\infty} a_n x^n = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right)^2$? *[2]*

SOLUTION. Since

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right)^2 = (e^x)^2 = e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n,$$

we must have $a_n = \frac{2^n}{n!}$ for all $n \geq 0$. □

4. Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges no matter what the value of x is. *[4]*

HINT. If x is negative, the series is an alternating series; if x is positive, you can, among other possibilities, use the Monotone Convergence Theorem.

SOLUTION. We will consider three cases, in order of complexity.

Case 1. ($x = 0$) As noted in the solution to question **2**, $\sum_{n=0}^{\infty} \frac{0^n}{n!} = 1 + 0 + 0 + \cdots$, which converges to 1 because every partial sum is equal to 1.

Case 2. ($x < 0$) Observe that if x is negative, then the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ alternates because even powers of x are positive and odd powers of x are negative. We know from question **5** on Assignment #4 that an alternating series $\sum_{n=0}^{\infty} a_n$ converges if $|a_{n+1}| < |a_n|$ for all n (past some point) and and also $\lim_{n \to \infty} |a_n| = 0$. The given series satisfies both of these conditions.

First, once $n + 1 > |x|$, we have $\left| \frac{x^{n+1}}{(n+1)!} \right| = \left| \frac{x}{n+1} \right| \cdot \left| \frac{x^n}{n!} \right| < \left| \frac{x^n}{n!} \right|$, *i.e.* past some point, the absolute value of the $n+1$st term in the series is less than the absolute value of the nth term, as required.

Second, let N be an integer greater than $|x|$. Then for $n > N$, $0 \leq \left|\frac{x^n}{n!}\right| = \left|\frac{x^{n-N}}{n(n-1)\cdots(N+1)}\right| \cdot \left|\frac{x^N}{N!}\right| < \left|\frac{x^{n-N}}{(N+1)^{n-N}}\right| \cdot \left|\frac{x^N}{N!}\right| = \left|\frac{x}{N}\right|^{n-N} \cdot \left|\frac{x^N}{N!}\right|$. Since $\lim_{n\to\infty} \left|\frac{x}{N}\right|^{n-N} \cdot \left|\frac{x^N}{N!}\right| = 0$ because $\left|\frac{x}{N}\right| < 1$, and $\lim_{n\to\infty} 0 = 0$, it follows by the Squeeze Theorem that $\lim_{n\to\infty} \left|\frac{x^n}{n!}\right| = 0$ as well.

Thus the given series converges by the Alternating Series Test when x is negative.

Case 3. ($x > 0$) For each $k \geq 0$, let $S_k = \sum_{n=0}^{k} \frac{x^n}{n!}$ be the kth partial sum of the given series. When $x > 0$, $\frac{x^{k+1}}{(k+1)!} > 0$, so $S_k < S_k + \frac{x^{k+1}}{(k+1)!} = S_{k+1}$ for all $k \geq 0$. That is, S_k is an increasing sequence when $x > 0$, and must therefore converge by the Monotone Convergence Theorem if we can find an upper bound for all the S_k.

As in Case 2 above, let N be an integer greater than $|x|$. Then, similarly to Case 2, when $n > N$, $\frac{x^n}{n!} = \frac{x^{n-N}}{n(n-1)\cdots(N+1)} \cdot \frac{x^N}{N!} < \frac{x^{n-N}}{(N+1)^{n-N}} \cdot \frac{x^N}{N!} = \left(\frac{x}{N}\right)^{n-N} \cdot \frac{x^N}{N!}$. It follows that when $n > N$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{N} \frac{x^n}{n!} + \sum_{n=N+1}^{\infty} \frac{x^n}{n!} &< S_N + \sum_{N+1}^{\infty} \left(\frac{x}{N}\right)^{n-N} \cdot \frac{x^N}{N!} \\
&= S_N + \frac{x^N}{N!} \cdot \left(\frac{x}{N}\right)^{-N} \sum_{n=N+1}^{\infty} \left(\frac{x}{N}\right)^n \\
&= S_N + \frac{x^N}{N!} \cdot \left(\frac{N}{x}\right)^{N} \sum_{n=N+1}^{\infty} \left(\frac{x}{N}\right)^n \\
&= S_N + \frac{N^N}{N!} \sum_{n=N+1}^{\infty} \left(\frac{x}{N}\right)^n
\end{aligned}$$

Note that $\sum_{n=N+1}^{\infty} \left(\frac{x}{N}\right)^n$ is a geometric series with first term $a = \left(\frac{x}{N}\right)^{N+1}$ and common ratio $r = \frac{x}{N}$, with $0 < r = \frac{x}{N} < 1$ since $0 < x < N$. This geometric series thus has a sum, namely $\frac{a}{1-r} = \frac{(x/N)^{N+1}}{1-x/N}$. It follows that for all $k \geq 0$,

$$S_k = \sum_{n=0}^{k} \frac{x^n}{n!} < \sum_{n=0}^{\infty} \frac{x^n}{n!} < S_N + \frac{N^N}{N!} \cdot \frac{(x/N)^{N+1}}{1-x/N},$$

i.e. $S_N + \frac{N^N}{N!} \cdot \frac{(x/N)^{N+1}}{1-x/N}$ is an upper bound for S_k.

Thus the given series converges by the Monotone Convergence Theorem. ■