Mathematics 1121H – Calculus II
Trent University, Winter 2026

Solutions to Assignment #6
Series Business II
Due on Friday, 27 February.

This assignment is concerned with the non-alternating counterparts of the series considered in Assigment #4, which work out differently.

First, consider the *harmonic series*, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$. This series does not *converge*, *i.e.* add up to a real number, which in this case means that it *diverges*, *i.e.* adds up to infinity. Despite its failure to add up to a real numer, this series and its partial sums arise in many areas of mathematics, enough so that the partial sums have their own name, the *harmonic numbers*, and notation $\displaystyle H_n = \sum_{k=1}^{n} \frac{1}{k}$.

1. Use a suitable `while` loop in SageMath to discover the least value of n required to ensure that $\displaystyle H_n = \sum_{k=1}^{n} \frac{1}{k} \geq a$ for each of $a = 3, 6, 9, 12$. *[2]*

Solution. Note the use of a `for` loop to deliver the successive values of a to the `while` loop.

```
[1]: # 1
clear_vars()
var('k')
var('a')
var('H_n')
H_n = 0
k = 0
for a in [3,6,9,12]:
    while( H_n < a ):
        k = k+1
        H_n = H_n + 1/k
    print( "a =", a, "least n= ", k, "H_n =",  N(H_n) )
    H_n = 0
    k = 0
```

```
a = 3 least n=  11 H_n = 3.01987734487735
a = 6 least n=  227 H_n = 6.00436670834557
a = 9 least n=  4550 H_n = 9.00020806293114
a = 12 least n=  91380 H_n = 12.0000030516656
```

□

2. Give an informal proof that the harmonic series diverges. *[2]*

SOLUTION. Here we go:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \cdots \\
&= [1] + \left[\frac{1}{2}\right] + \left[\frac{1}{3} + \frac{1}{4}\right] + \left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] + \left[\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right] + \cdots \\
&\geq \left[\frac{1}{2}\right] + \left[\frac{1}{2}\right] + \left[2 \cdot \frac{1}{4}\right] + \left[4 \cdot \frac{1}{8}\right] + \left[8 \cdot \frac{1}{16}\right] + \cdots \\
&= \left[\frac{1}{2}\right] + \left[\frac{1}{2}\right] + \left[\frac{1}{2}\right] + \left[\frac{1}{2}\right] + \cdots = \sum_{k=1}^{\infty} \frac{1}{2} = \infty
\end{aligned}$$

A series can add up to at least infinity only if it adds up to infinity ... □

Second, consider the series that adds up the reciprocals of all the squares of positive integers, $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$. (This series seems to lack a commonly accepted name; your instructor likes to call it the *square harmonic series*.) Unlike the harmonic series, it is convergent: it sums to $\frac{\pi^2}{6}$, a fact first proved by Leonhard Euler (1707-1783) in his solution to what came to be called the *Basel problem*.

3. Use a suitable `while` loop in SageMath to discover the least value of n required to ensure that $\sum_{k=1}^{n} \frac{1}{k^2} \geq \frac{\pi^2}{6} - \frac{1}{a}$ for each of $a = 5$, 40, 100, 1000. *[2]*

SOLUTION. Here we are:

[2]:
```
# 3
clear_vars()
var('k')
var('a')
var('partial_sum')
var('full_sum')
full_sum = pi^2/6
print( "full sum =", full_sum, "=", N(full_sum) )
partial_sum = 0
k = 0
for a in [5,40,100,1000]:
    while( partial_sum < full_sum - 1/a ):
        k = k+1
        partial_sum = partial_sum + 1/k^2
    print( "a =", a, "least n= ", k, "partial sum =",  N(partial_sum) )
    partial_sum = 0
    k = 0
```

```
full sum = 1/6*pi^2 = 1.64493406684823
a = 5 least n=  5 partial sum = 1.46361111111111
a = 40 least n=  40 partial sum = 1.62024396300694
a = 100 least n=  100 partial sum = 1.63498390018489
a = 1000 least n=  1000 partial sum = 1.64393456668156
```
□

4. What pattern do you see in your solution to question **3**? Does this pattern hold for all positive integers a? If so, try to explain why; if not, give an example where the pattern fails. *[2]*

HINT. At minimum, some experimentation with other values of a besides those asked for in question **3** is in order. Probably some research of the "look it up" variety, too.

SOLUTION. The pattern is pretty obvious: the least n such that $\sum_{k=1}^{n} \frac{1}{k^2} \geq \frac{\pi^2}{6} - \frac{1}{a}$ for a given integer a is $n = a$. Some experimentation with the code in the solution to question **3** above to check what happens for a from 1 to 20 showed that the pattern held for these values of a.

Since $\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}$, the inequality $\sum_{k=1}^{n} \frac{1}{k^2} \geq \frac{\pi^2}{6} - \frac{1}{a}$ is equivalent to saying that

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \leq \frac{1}{a}.$$

That is, one form of the pattern is that the least n such that $\sum_{k=n+1}^{\infty} \frac{1}{k^2} \leq \frac{1}{a}$ is $n = a$.

This pattern is true because for all $n \geq 1$, $\frac{1}{n+1} < \sum_{k=n+1}^{\infty} \frac{1}{k^2} < \frac{1}{n}$. This guarantees, on the one hand, that $\sum_{k=n+1}^{\infty} \frac{1}{k^2} < \frac{1}{n} = \frac{1}{a}$ for $n = a$, and on the other hand, that $n = a$ gives the first such sum below $\frac{1}{a}$ because $\sum_{k=(n-1)+1}^{\infty} \frac{1}{k^2} > \frac{1}{(n-1)+1} = \frac{1}{n} = \frac{1}{a}$ when $n = a$.

It remains to show that for all $n \geq 1$, $\frac{1}{n+1} < \sum_{k=n+1}^{\infty} \frac{1}{k^2} < \frac{1}{n}$. We will prove each of the inequalities separately using essentially the same algebraic tricks. First, if $b < c$, then $\frac{1}{c} < \frac{1}{b}$. Second, as long as $b \neq 0$ and $b \neq -1$, $\frac{1}{b} - \frac{1}{b+1} = \frac{b+1-b}{b(b+1)} = \frac{1}{b(b+1)}$. For the left inequality:

$$\begin{aligned}
\frac{1}{n+1} &= \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} + \frac{1}{n+3} - \frac{1}{n+4} + \frac{1}{n+4} - \cdots \\
&= \left[\frac{1}{n+1} - \frac{1}{n+2}\right] + \left[\frac{1}{n+2} - \frac{1}{n+3}\right] + \left[\frac{1}{n+3} - \frac{1}{n+4}\right] + \cdots \\
&= \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+3)(n+4)} + \frac{1}{(n+4)(n+5)} + \cdots \\
&< \frac{1}{(n+1)(n+1)} + \frac{1}{(n+2)(n+2)} + \frac{1}{(n+3)(n+3)} + \frac{1}{(n+4)(n+4)} + \cdots \\
&= \sum_{k=n+1}^{\infty} \frac{1}{k^2}
\end{aligned}$$

For the right inequality:

$$\begin{aligned}
\sum_{k=n+1}^{\infty} \frac{1}{k^2} &= \frac{1}{(n+1)(n+1)} + \frac{1}{(n+2)(n+2)} + \frac{1}{(n+3)(n+3)} + \frac{1}{(n+4)(n+4)} + \cdots \\
&< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+3)(n+4)} + \cdots \\
&= \left[\frac{1}{n} - \frac{1}{n+1}\right] + \left[\frac{1}{n+1} - \frac{1}{n+2}\right] + \left[\frac{1}{n+2} - \frac{1}{n+3}\right] + \cdots \\
&= \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} + \frac{1}{n+3} - \cdots \\
&= \frac{1}{n}
\end{aligned}$$

Whew! □

5. Given that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, what does $\displaystyle\sum_{i=1}^{\infty} \frac{1}{(2i-1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots$ add up to? Explain why. *[2]*

Solution. Here we go:

$$\begin{aligned}
\frac{\pi^2}{6} &= \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = 1 + \frac{1}{9} + \frac{1}{25} + \cdots + \frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \cdots \\
&= \sum_{i=1}^{\infty} \frac{1}{(2i-1)^2} + \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots\right) = \sum_{i=1}^{\infty} \frac{1}{(2i-1)^2} + \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} \\
\implies &\sum_{i=1}^{\infty} \frac{1}{(2i-1)^2} = \frac{3}{4} \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{3}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8} \quad \square
\end{aligned}$$