Trent University, Summer 2018

Mathematics 1120H – Calculus II: Integrals and Series

MATH 1120H Test Solutions

Monday, 9 July

Time: 50 minutes

Instructions

- *Show all your work.* Legibly, please! Simplify where you reasonably can.
- *If you have a question, ask it!*
- Use the back sides of all the pages for rough work or extra space.
- You may use a calculator and (all sides of) an aid sheet.

1. Compute any *four* (4) of integrals **a**–**f**. *[12 = 4 × 3 each]*

a. $\displaystyle\int \tan^3(x)\sec^3(x)\,dx$ **b.** $\displaystyle\int_0^1 \frac{1}{\sqrt{y}}\,dy$ **c.** $\displaystyle\int \frac{z^2-1}{z^2+2z+1}\,dz$

d. $\displaystyle\int_1^e t\ln(t)\,dt$ **e.** $\displaystyle\int s^2e^s\,ds$ **f.** $\displaystyle\int_0^3 \frac{r^2}{r^3+9}\,dr$

Solutions. **a.** We will use the trigonometric identity $\tan^2(x) = \sec^2(x) + 1$, followed by the substitution $u = \sec(x)$, so $du = \sec(x)\tan(x)\,dx$.

$$\begin{aligned}\int \tan^3(x)\sec^3(x)\,dx &= \int \tan^2(x)\sec^2(x)\sec(x)\tan(x)\,dx\\ &= \int \left(\sec^2(x)-1\right)\sec^2(x)\sec(x)\tan(x)\,dx = \int \left(u^2-1\right)u^2\,du\\ &= \int \left(u^4-u^2\right)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C\\ &= \frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C \quad \square\end{aligned}$$

b. $\displaystyle\int_0^1 \frac{1}{\sqrt{y}}\,dy = \int_0^1 y^{-1/2}\,dy = \left.\frac{y^{1/2}}{1/2}\right|_0^1 = 2\sqrt{y}\Big|_0^1 = 2\sqrt{1}-2\sqrt{0} = 2\cdot 1 - 2\cdot 0 = 2-0 = 2 \quad \square$

Note. The integral above is technically an indefinite integral because $\frac{1}{\sqrt{y}}$ has an asymptote at $y = 0$, but this is one of the cases where one gets away with ignoring that fact because the antiderivative does not have an asymptote at 0.

c. Algebra is our friend! We will also use the trivial substitution $w = z + 1$, so $dw = dz$.

$$\begin{aligned}\int \frac{z^2-1}{z^2+2z+1}\,dz &= \int \frac{(z-1)(z+1)}{(z+1)^2}\,dz = \int \frac{z-1}{z+1}\,dz = \int \frac{(z+1)-2}{z+1}\,dz\\ &= \int \left(\frac{z+1}{z+1} - \frac{2}{z+1}\right)\,dz = \int 1\,dz - 2\int \frac{1}{z+1}\,dz\\ &= z - 2\int \frac{1}{w}\,dw = z - 2\ln(w) + C = z - 2\ln(z+1) + C \quad \square\end{aligned}$$

d. We will use integration by parts with $u = \ln(t)$ and $v' = t$, so $u' = \dfrac{1}{t}$ and $v = \dfrac{t^2}{2}$.

$$\int_1^e t\ln(t)\,dt = \frac{t^2}{2}\ln(t)\Big|_1^e - \int_1^e \frac{1}{t}\cdot\frac{t^2}{2}\,dt = \frac{e^2}{2}\ln(e) - \frac{1^2}{2}\ln(1) - \frac{1}{2}\int_1^e t\,dt$$
$$= \frac{e^2}{2}\cdot 1 - \frac{1}{2}\cdot 0 - \frac{1}{2}\cdot\frac{t^2}{2}\Big|_1^e = \frac{e^2}{2} - 0 - \left[\frac{e^2}{4} - \frac{1^2}{4}\right] = \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4} \quad \square$$

e. We will use integration by parts twice. The first will have $u = s^2$ and $v' = e^s$, so $u' = 2s$ and $v = e^s$; the second will have $x = 2s$ and $y' = e^s$, so $x' = 2$ and $y = e^s$.

$$\int s^2e^s\,ds = s^2e^s - \int 2se^s\,ds = s^2e^s - \left[2se^s - \int 2e^s\,ds\right] = s^2e^s - 2se^s + 2e^s + C \quad \square$$

f. We will use the substitution $w = r^3 + 9$, so $dw = 3r^2\,dr$ and $r^2\,dr = \frac{1}{3}\,dw$, and change the limits as we go along: $\begin{array}{ccc} r & 0 & 3 \\ w & 9 & 36 \end{array}$.

$$\int_0^3 \frac{r^2}{r^3+9}\,dr = \int_9^{36} \frac{1}{w}\cdot\frac{1}{3}\,dw = \frac{\ln(w)}{3}\Big|_9^{36} = \frac{\ln(36)}{3} - \frac{\ln(9)}{3} = \frac{\ln\left(\frac{36}{9}\right)}{3} = \frac{\ln(4)}{3} \quad \blacksquare$$

2. Do any *two* (2) of parts **a**–**c**. *[8 = 2 × 4 each]*

- **a.** Use a Right-Hand Rule sum to approximate $\displaystyle\int_0^2 2x\,dx$, ensuring that it is within 1 of the exact value.
- **b.** Find the area between the curves $y = \cos(x)$ and $y = \sin(x)$ for $0 \le x \le \pi$.
- **c.** Compute $\displaystyle\int \sqrt{t}\cdot e^{\sqrt{t}}\,dt$.

SOLUTIONS. **a.** If we let $f(x) = 2x$, then $|f'(x)| = |2| = 2$ for all $x \in [0,2]$. We know from class that the difference between the Right-Hand Rule sum for n and the definite integral $\int_a^b f(x)\,dx$ it approximates is at most $M(b-a)^2/n$. In this case we have $a = 0$ and $b = 2$, and if we let $f(x) = 2x$, then $|f'(x)| = |2| = 2$ for all $x \in [0,2]$, so we can use $M = 2$. It follows that we need to pick an n such that $2(2-0)^2/n = 8/n \le 1$. To make this happen we need to have $n \ge 8$; to minimize the amount of arithmetic we need to do, we will use $n = 8$. Then:

$$\int_0^2 2x\,dx \approx \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) = \frac{2-0}{8}\sum_{i=1}^{8} 2\left(0 + i\frac{2-0}{8}\right) = \frac{1}{4}\sum_{i=1}^{8}\frac{4i}{8}$$
$$\frac{1}{4}\sum_{i=1}^{8}\frac{1}{2}i = \frac{1}{8}\sum_{i=1}^{8} i = \frac{1}{8}(1+2+3+4+5+6+7+8) = \frac{36}{8} = \frac{9}{2} = 4.5$$

This is guaranteed to be within 1 of the exact value of the integral. As a check, since the integral is very easy to evaluate, we do so. $\displaystyle\int_0^2 2x\,dx = x^2\Big|_0^2 = 2^2 - 0^2 = 4$, so our approximation above is indeed within 1 of the correct value. $\square$

b. $\cos(0) = 1$ and $\sin(0) = 0$ while $\cos(\pi) = -1$ and $\sin(0) = 0$. Between 0 and π, we have one point where $\cos(x) = \sin(x)$, namely $x = \frac{\pi}{4}$, where $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. It follows that $\cos(x) \geq \sin(x)$ for $0 \leq x \leq \frac{\pi}{4}$ and $\cos(x) \leq sin(x)$ for $\frac{\pi}{4} \leq x \leq \pi$, and so the area between the curves for $0 \leq x \leq \pi$ is given by:

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/4} [\cos(x) - \sin(x)]\ dx + \int_{\pi/4}^{\pi} [\sin(x) - \cos(x)]\ dx \\
&= [\sin(x) - (-\cos(x))]_0^{\pi/4} + [(-\cos(x)) - \sin(x)]_{\pi/4}^{\pi} \\
&= \left[\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\right] - [\sin(0) + \cos(0)] \\
&\quad + [-\cos(\pi) - \sin(\pi)] - \left[-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right] \\
&= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] - [0 + 1] + [-(-1) - 0] - \left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] \\
&= \frac{4}{\sqrt{2}} = 2\sqrt{2} \quad \square
\end{aligned}$$

c. We will use the substitution $t = s^2$, so $dt = 2s\,du$; note that $s = \sqrt{t}$. Then

$$\int \sqrt{t} \cdot e^{\sqrt{t}}\,dt = \int se^s\,2s\,ds = 2\int s^2 e^s\,ds\,,$$

which is twice the integral in **1e**, so, by the calculation done in that solution,

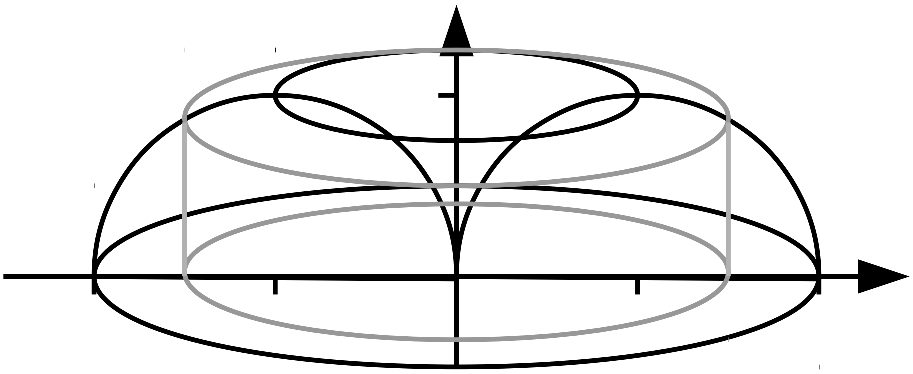
$$\int \sqrt{t} \cdot e^{\sqrt{t}}\,dt = 2\left[s^2e^s - 2se^s + 2e^s\right] + C = 2te^{\sqrt{t}} - 4\sqrt{t}e^{\sqrt{t}} + 4e^{\sqrt{t}} + C\,. \quad \blacksquare$$

3. Do either *one* (1) of parts **a** or **b**. *[10]*

- **a.** The region in the first quadrant between the parabola $y = 2x - x^2$ and the x-axis is rotated all the way about the y-axis. Find the volume of the resulting solid.
- **b.** A truncated pyramid is 50 m tall, has a square base with sides of length 100 m, and a square top with sides of length 50 m parallel to the base. Find the volume of the pyramid.

SOLUTIONS. **a.** Note that $y = 2x - x^2 = x(x - 2)$ is a parabola that opens downwards and has x-intercepts at $x = 0$ and $x = 2$.

We will use the method of cylindrical shells ("nested dolls") to find the volume of the given solid of revolution. Since the axis of revolution is the y-axis, the cylindrical shell at



x, for $0 \le x \le 2$, has radius $r = x - 0 = x$ and height $h = y - 0 = y = 2x - x^2$, and hence area $A(x) = 2\pi r h = 2\pi x \left(2x - x^2\right) = 2\pi \left(2x^2 - x^3\right)$. It follows that the volume of the solid is given by:

$$V = \int_0^2 A(x)\,dx = \int_0^2 2\pi \left(2x^2 - x^3\right)\,dx = 2\pi \left(\frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^2$$
$$= 2\pi \left(\frac{2}{3}2^3 - \frac{1}{4}2^4\right) - 2\pi \left(\frac{2}{3}0^3 - \frac{1}{4}0x^4\right) = 2\pi \left(\frac{16}{3} - 4\right) - 0 = \frac{8}{3}\pi \quad \square$$

NOTE. Part **a** can also be done using the disk/washer method, but the algebra needed to find the radii involved is messier, as is the resulting integral.

b. The cross-section at height $h = 0$ has side length $s(0) = 100$ and the cross-section at height $h = 50$ has side length $s(50) = 50$. Consider the function $s(h)$. First, since pyramids have straight sides, it ought to be linear. Second, it must then have slope $\frac{50-100}{50-0} = \frac{-50}{50} = -1$, and we already know that $s(0) = 100$. Thus $s(h) = -h + 100$, and the square cross-section at height h, for $0 \le h \le 50$, has area $A(h) = [s(h)]^2 = [-h + 100]^2 = h^2 - 200h + 10000$. The volume of the truncated pyramid is therefore:

$$V = \int_0^{50} A(h)\,dh = \int_0^{50} \left(h^2 - 200h + 10000\right)\,dh = \left(\frac{1}{3}h^3 - \frac{200}{2}h^2 + 10000h\right)\Big|_0^5 0$$
$$= \left(\frac{1}{3}\cdot 50^3 - 100\cdot 50^2 + 10000\cdot 50\right) - \left(\frac{1}{3}\cdot 0^3 - 100\cdot 0^2 + 10000\cdot 0\right)$$
$$= \frac{125000}{3} - 250000 + 500000 - 0 = \frac{7}{3}\cdot 125000 = 291666.\dot{6}\ m^3 \quad \blacksquare$$

Minor corrections made 2018.07.11.

[Total = 30]