Mathematics 1120H – Calculus I: Integrals and Series
TRENT UNIVERSITY, Summer 2018
Solutions to the Practice Final Examination

Time: *3 hours.* *Brought to you by* Стефан Біланюк.

Instructions: Do parts **A**, **B**, and **C**, and, if you wish, part **D**. Show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: Any calculator; (all sides of) one aid sheet; one (1) brain (no neuron limit).

Part A. Do all four (4) of **1**–**4**.

1. Evaluate any *four* (4) of the integrals **a**–**f**. *[20 = 4 × 5 each]*

$$\textbf{a. } \int z\cos(2z)\,dz \qquad \textbf{b. } \int_0^1 te^{-t^2}\,dt \qquad \textbf{c. } \int \frac{x+1}{x^2+1}\,dx$$

$$\textbf{d. } \int_{-1}^1 \frac{1}{\sqrt{y^2+1}}\,dy \qquad \textbf{e. } \int \frac{s^2}{s^2-1}\,ds \qquad \textbf{f. } \int_0^{\pi/4} \frac{\sin^3(w)}{\cos^2(w)}\,dw$$

SOLUTIONS. **a.** We will use integration by parts with $u = z$ and $v' = \cos(2z)$, so $u' = 1$ and $v = \frac{1}{2}\sin(2z)$.

$$\begin{aligned}\int z\cos(2z)\,dz &= z \cdot \frac{1}{2}\sin(2z) - \int 1 \cdot \frac{1}{2}\sin(2z)\,dz = \frac{1}{2}z\sin(2z) - \frac{1}{2}\left(-\frac{1}{2}\cos(2z)\right) + C \\ &= \frac{1}{2}z\sin(2z) + \frac{1}{4}\cos(2z) + C \quad \square\end{aligned}$$

b. We wil use the substitution $u = -t^2$, so $du = -2t\,dt$ and $t\,dt = \left(-\frac{1}{2}\right)du$, while $\begin{array}{ccc} x & 0 & 1 \\ u & 0 & -1 \end{array}$.

$$\begin{aligned}\int_0^1 te^{-t^2}\,dt &= \int_0^{-1} e^u\left(-\frac{1}{2}\right)du = \frac{1}{2}\int_{-1}^0 e^u\,du \\ &= \frac{1}{2}e^u\Big|_{-1}^0 = \frac{1}{2}e^0 - \frac{1}{2}e^{-1} = \frac{1}{2}\left(1 - \frac{1}{e}\right) \quad \square\end{aligned}$$

c. We will use a little cheap algebra and the substitution $w = x^2 + 1$, so $dw = 2x\,dx$ and $x\,dx = \frac{1}{2}\,dw$.

$$\begin{aligned}\int \frac{x+1}{x^2+1}\,dx &= \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx = \int \frac{1}{u}\cdot\frac{1}{2}\,du + \arctan(x) \\ &= \frac{1}{2}\ln(u) + \arctan(x) + C = \frac{1}{2}\ln\left(x^2+1\right) + \arctan(x) + C \quad \square\end{aligned}$$

d. We will use the trigonometric substitution $y = \tan(\theta)$, so $dy = \sec^2(\theta)\, d\theta$ while $\begin{array}{ccc} y & -1 & 1 \\ \theta & -\pi/4 & \pi/4 \end{array}$. We will also use the facts that $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, so that $\tan\left(\frac{\pi}{4}\right) = 1$, $\tan\left(-\frac{\pi}{4}\right) = -1$, and $\sec\left(\frac{\pi}{4}\right) = \sec\left(-\frac{\pi}{4}\right) = \sqrt{2}$.

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{\sqrt{y^2+1}}\, dy &= \int_{-\pi/4}^{\pi/4} \frac{1}{\sqrt{\tan^2(\theta)+1}} \sec^2(\theta)\, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{\sqrt{\sec^2(\theta)}} \sec^2(\theta)\, d\theta \\
&= \int_{-\pi/4}^{\pi/4} \sec(\theta)\, d\theta = \ln\left(\sec(\theta) + \tan(\theta)\right)\Big|_{-\pi/4}^{\pi/4} \\
&= \ln\left(\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right) - \ln\left(\sec\left(-\frac{\pi}{4}\right) + \tan\left(-\frac{\pi}{4}\right)\right) \\
&= \ln\left(\sqrt{2}+1\right) - \ln\left(\sqrt{2}-1\right) \quad \square
\end{aligned}$$

e. We will use a little algebra and partial fractions. First, note that:

$$\frac{s^2}{s^2-1} = \frac{s^2-1+1}{s^2-1} = \frac{s^2-1}{s^2-1} + \frac{1}{s^2-1} = 1 + \frac{1}{(s-1)(s+1)}$$

Second,

$$\frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} = \frac{A(s+1)}{(s-1)(s+1)} + \frac{B(s-1}{(s-1)(s+1)} = \frac{(A+B)s + (A-B)}{(s-1)(s+1)}$$

for some constants A and B. Since we must have $A+B=0$ and $A-B=1$, it follows from adding these two equations that $2A = 1$, *i.e.* $A = \frac{1}{2}$, and then substituting into either equation and solving for B gives $B = -\frac{1}{2}$. Thus:

$$\begin{aligned}
\int \frac{s^2}{s^2-1}\, ds &= \int \left(1 + \frac{1}{(s-1)(s+1)}\right) ds = \int 1\, ds + \int \frac{1}{(s-1)(s+1)}\, ds \\
&= s + \int \left(\frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}}{s+1}\right) ds = s + \frac{1}{2}\int \frac{1}{s-1}\, ds - \frac{1}{2}\int \frac{1}{s+1}\, ds \\
&= s + \frac{1}{2}\ln(s-1) - \frac{1}{2}\ln(s+1) + C \quad \square
\end{aligned}$$

f. We will use the trigonometric identity $\cos^2(w) + \sin^2(w) = 1$, a bit of algebra, and the substitution $x = \cos(w)$, so $dx = -\sin(w)\, dw$ and $\sin(w)\, dw = (-1)\, dx$, while $\begin{array}{ccc} w & 0 & \pi/4 \\ x & 1 & \frac{1}{\sqrt{2}} \end{array}$.

$$\begin{aligned}
\int_0^{\pi/4} \frac{\sin^3(w)}{\cos^2(w)}\, dw &= \int_0^{\pi/4} \frac{\left(1-\cos^2(w)\right)\sin(w)}{\cos^2(w)}\, dw = \int_1^{1/\sqrt{2}} \frac{1-x^2}{x^2}\,(-1)\, dx \\
&= \int_{1/\sqrt{2}}^{1} \left(x^{-2}-1\right)\, dx = \left(-x^{-1}-x\right)\Big|_{1/\sqrt{2}}^{1} = \left(-\frac{1}{x} - x\right)\Bigg|_{1/\sqrt{2}}^{1} \\
&= \left(-\frac{1}{1} - 1\right) - \left(-\frac{1}{1/\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = -2 + \sqrt{2} + \frac{1}{\sqrt{2}} \quad \blacksquare
\end{aligned}$$

2. Determine whether the series converges in any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

$$\textbf{a.}\ \sum_{n=0}^{\infty} \frac{n^2}{2^n} \qquad \textbf{b.}\ \sum_{m=1}^{\infty} \frac{\sin(m\pi)}{\ln(m\pi)} \qquad \textbf{c.}\ \sum_{\ell=2}^{\infty} e^{-\ell^2}$$

$$\textbf{d.}\ \sum_{k=3}^{\infty} \frac{k! \cdot 2^k}{3^k} \qquad \textbf{e.}\ \sum_{j=4}^{\infty} \frac{j^2 - j + 1}{\sqrt{j^5 + 13}} \qquad \textbf{f.}\ \sum_{i=5}^{\infty} \cos(i\pi)\sqrt{\left(\frac{1}{2}\right)^i}$$

SOLUTIONS. **a.** We will use the Ratio Test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{n^2 + 2n + 1}{n^2} \cdot \frac{2^n}{2^{n+1}}$$
$$= \lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \cdot \frac{1}{2} = (1 + 0 + 0) \cdot \frac{1}{2} = \frac{1}{2} < 1$$

It follows by the Ratio Test that $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$ converges. □

b. This is a bit of a trick question: $\sin(m\pi) = 0$ for all integers m, so the series is just $\sum_{m=1}^{\infty} 0$, which certainly converges. □

c. Since $0 < \frac{1}{e} < 1$ and $0 < e^{-\ell^2} = \frac{1}{e^{\ell^2}} = \left(\frac{1}{e}\right)^{\ell^2} < \left(\frac{1}{e}\right)^{\ell}$ whenever $\ell \geq 2$, the given series converges by comparison with the geometric series $\sum_{\ell=2}^{\infty} \left(\frac{1}{e}\right)^{\ell}$. □

d. We will use the Ratio Test. Since

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{\frac{(k+1)! \cdot 2^{k+1}}{3^{k+1}}}{\frac{k! \cdot 2^k}{3^k}}\right| = \lim_{k\to\infty} \frac{(k+1)! \cdot 2^{k+1}}{3^{k+1}} \cdot \frac{3^k}{k! \cdot 2^k}$$
$$= \lim_{k\to\infty} \frac{2(k+1)}{3} = \infty > 1\,,$$

the given series does not converge. □

e. Looking at the dominant terms in $\frac{j^2 - j + 1}{\sqrt{j^5 + 13}}$ suggests that the given series should converge, or not, as $\sum_{j=4}^{\infty} \frac{j^2}{\sqrt{j^5}}$ does. Since

$$\lim_{j\to\infty} \frac{\frac{j^2 - j + 1}{\sqrt{j^5 + 13}}}{\frac{j^2}{\sqrt{j^5}}} = \lim_{j\to\infty} \frac{\left(j^2 - j + 1\right)/j^2}{\left(\sqrt{j^5 + 13}\right)/\sqrt{j^5}} = \lim_{j\to\infty} \frac{1 - \frac{1}{j} + \frac{1}{j^2}}{\sqrt{1 + \frac{13}{j^5}}} = \frac{1 - 0 + 0}{\sqrt{1 + 0}} = \frac{1}{1} = 1\,,$$

the Limit Comparison Test tells us that $\displaystyle\sum_{j=4}^{\infty} \frac{j^2-j+1}{\sqrt{j^5+13}}$ and $\displaystyle\sum_{j=4}^{\infty} \frac{j^2}{\sqrt{j^5}}$ do indeed both converge or both diverge. Since $\displaystyle\sum_{j=4}^{\infty} \frac{j^2}{\sqrt{j^5}} = \sum_{j=4}^{\infty} \frac{j^2}{j^{5/2}} = \sum_{j=4}^{\infty} \frac{1}{j^{1/2}}$ diverges by the p-Test, as $p = \frac{1}{2} - 0 = \frac{1}{2} \leq 1$, this means that $\displaystyle\sum_{j=4}^{\infty} \frac{j^2-j+1}{\sqrt{j^5+13}}$ also diverges. $\square$

f. $\displaystyle\sum_{i=5}^{\infty} \cos(i\pi)\sqrt{\left(\frac{1}{2}\right)^i} = \sum_{i=5}^{\infty}(-1)^i\left(\frac{1}{\sqrt{2}}\right)^i = \sum_{i=5}^{\infty}\left(-\frac{1}{\sqrt{2}}\right)^i$ converges because it is a geometric series with common ratio $r = -\dfrac{1}{\sqrt{2}}$ and $|r| = \dfrac{1}{\sqrt{2}} < 1$. $\blacksquare$

3. Do any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. Use the Right-Hand Rule or the Trapezoid Rule to approximate $\displaystyle\int_0^1 \left(1-x^2\right)\,dx$ to within $\frac{1}{2} = 0.5$ of the exact value.

b. Find the area of the finite region between $y = x^2$ and $y = x + 2$.

c. Suppose $a_1 = 1$ and $a_{n+1} = \dfrac{n+1}{n} a_n$. Compute $\displaystyle\lim_{n\to\infty} a_n$.

d. Find the volume of the solid obtained by revolving the region below $y = 2$ and above $y = 1$, for $1 \leq x \leq 2$, about the y-axis.

e. Suppose $\sigma(n) = \begin{cases} 1 & \text{if } n = 4k \text{ or } 4k+1 \text{ for some integer } k \\ -1 & \text{if } n = 4k+2 \text{ or } 4k+3 \text{ for some integer } k \end{cases}$. What function has $\displaystyle\sum_{n=0}^{\infty} \frac{\sigma(n)x^n}{n!}$ as its Taylor series at $a = 0$?

f. Find the Taylor series at $a = 0$ of $f(x) = e^{2x}$ and determine its interval of convergence.

SOLUTIONS. **a.** *(Right-Hand Rule)* Let $f(x) = 1 - x^2$; then $f'(x) = -2x$ and it is easy to see that $|f'(x)| = |-2x| = 2|x| \leq 2$ for all $x \in [0, 1]$. We know from class that the difference between the Right-Hand Rule sum for n and the definite integral $\int_a^b f(x)\,dx$ it approximates is at most $M(b-a)^2/n$, where M is an upper bound for $|f'(x)|$ for $x \in [a, b]$. In this case $a = 0$, $b = 1$, and we can let $M = 2$. We need to choose n to ensure that $2(1-0)^2/n = 2/n \leq 0.5$, which is equivalent to asking that $n \geq 2/0.5 = 4$. The Right-Hand

Rule sum for $\int_0^1 \left(1-x^2\right)\,dx$ with $n=4$ is:

$$\begin{aligned}\sum_{i=1}^{4} \frac{1-0}{4} f\left(0+i\frac{1-0}{4}\right) &= \frac{1}{4}\sum_{i=1}^{4} f\left(\frac{i}{4}\right) = \frac{1}{4}\sum_{i=1}^{4}\left(1-\left(\frac{i}{4}\right)^2\right) \\ &= \frac{1}{4}\left[\left(1-\frac{1}{16}\right)+\left(1-\frac{4}{16}\right)+\left(1-\frac{9}{16}\right)+\left(1-\frac{16}{16}\right)\right] \\ &= \frac{1}{4}\left[\frac{15}{16}+\frac{12}{16}+\frac{7}{16}+0\right] = \frac{1}{4}\cdot\frac{34}{16} = \frac{17}{32} \quad \square\end{aligned}$$

a. *(Trapezoid Rule)* Let $f(x)=1-x^2$; then $f'(x)=-2x$ and $f''(x)=-2$. It is easy to see that $|f''(x)|=|-2|=2$ for all $x\in[0,1]$. We know from class and the textbook that the difference between the Trapezoid Rule sum for n and the definite integral $\int_a^b f(x)\,dx$ it approximates is at most $\frac{M(b-a)^3}{n^2}$, where M is an upper bound for $|f'(x)|$ for $x\in[a,b]$. In this case $a=0$, $b=1$, and we can let $M=2$. We need to choose n to ensure that $2(1-0)^3/n^2 = 2/n^2 \leq 0.5$, which is equivalent to asking that $n^2 \geq 2/0.5 = 4$, *i.e.* that $n\geq 2$. The Trapezoid Rule sum for $\int_0^1 \left(1-x^2\right)\,dx$ with $n=2$ is:

$$\begin{aligned}&\frac{1-0}{2}\left[\frac{1}{2}f\left(0+0\frac{1-0}{2}\right)+f\left(0+1\frac{1-0}{2}\right)+\frac{1}{2}f\left(0+2\frac{1-0}{2}\right)\right] \\ &= \frac{1}{2}\left[\frac{1}{2}f(0)+f\left(\frac{1}{2}\right)+\frac{1}{2}f(1)\right] = \frac{1}{2}\left[\frac{1}{2}\left(1-0^2\right)+\left(1-\left(\frac{1}{2}\right)^2\right)+\frac{1}{2}\left(1-1^2\right)\right] \\ &= \frac{1}{2}\left[\frac{1}{2}+\frac{3}{4}+0\right] = \frac{1}{2}\cdot\frac{5}{4} = \frac{5}{8} \quad \square\end{aligned}$$

NOTE. Both of the values obtained above, $\frac{17}{32}$ using the Right-Hand Rule and $\frac{5}{8}$ using the Trapezoid Rule, are within 0.5 of the correct value of $\frac{2}{3}$ for $\int_0^1 \left(1-x^2\right)\,dx$.

b. We first need to find out where $y=x^2$ and $y=x+2$ cross. If $x^2=y=x+2$, then $0=x^2-x-2=(x+1)(x-2)$, so $x=-1$ or $x=2$. By comparing y values at $x=0$, $0^2=0<2=0+2$, we can see that $y=x+2$ is above $y=x^2$ for $-1<x<2$. It follows that the area of the region is:

$$\begin{aligned}\text{Area} &= \int_{-1}^{2}\left(x+2-x^2\right)\,dx = \left(\frac{1}{2}x^2+2x-\frac{1}{3}x^3\right)\Big|_{-1}^{2} \\ &= \left(\frac{1}{2}2^2+2\cdot 2-\frac{1}{3}2^3\right)-\left(\frac{1}{2}(-1)^2+2(-1)-\frac{1}{3}(-1)^3\right) \\ &= \frac{10}{3}-\left(-\frac{7}{6}\right) = \frac{27}{6} = \frac{9}{2} \quad \square\end{aligned}$$

c. Observe that if $n > 1$, then $a_n = \dfrac{n}{n-1} a_{n-1} = \dfrac{n}{n-1} \cdot \dfrac{n-1}{n-2} a_{n-2} = \dfrac{n}{n-2} a_{n-2} = \dfrac{n}{n-2} \cdot \dfrac{n-2}{n-3} a_{n-3} = \dfrac{n}{n-3} a_{n-3} = \cdots = \dfrac{n}{1} a_1 = \dfrac{n}{1} \cdot 1 = n$. Thus $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$. $\square$

d. *(Washers)* We are revolving the region about the y-axis, so if we use the disk/washer method to compute the volume, we should use y as the fundamental variable. In this case, $1 \leq y \leq 2$ for our region; the outside radius of the washer at y is the distance between the y-axis and the line $x = 2$, $R = 2$, and the inside radius of the washer at y is the distance between the y-axis and the line $x = 1$, $r = 1$. The washer at y thus has area $\pi R^2 - \pi r^2 = \pi 2^2 - \pi 1^2 = 4n - \pi = 3\pi$. It follows that the volume of the solid is:

$$\mathrm{V} = \int_1^2 A(y)\, dy = \int_1^2 3\pi\, dy = 3\pi y\big|_1^2 = 3\pi \cdot 2 - 3\pi \cdot 1 = 3\pi \quad \square$$

d. *(Shells)* We are revolving the region about the y-axis, so if we use the cylindrical shell method to compute the volume, we should use x as the fundamental variable. In this case, $1 \leq x \leq 2$ for our region; since we are rotating the region about the y-axis, the radius of the cylindrical shell at x is just $r = x$, and its height is the distance between $y = 2$ and $y = 1$, namely $h = 2 - 1 = 1$. The shell at x thus has area $A(x) = 2\pi r h = 2\pi x \cdot 1 = 2\pi x$. It follows that the volume of the solid is:

$$\mathrm{V} = \int_1^2 A(x)\, dx = \int_1^2 2\pi x\, dy = \pi x^2\big|_1^2 = \pi 2^2 - \pi 1^2 = 3\pi \quad \square$$

d. *(Geometry)* The solid in question is a cylinder of height 1 and radius 2 with a cylinder of height 1 and radius 1 cut out from it. The volume of a cylinder of height h are radius r is $\pi r^2 h$, so the volume of the given shape is $\pi 2^2 1 - \pi 1^2 1 = 4\pi - \pi = 3\pi$. $\square$

e. Note that the given series converges absolutely for all x by the Ratio Test because

$$\lim_{n\to\infty} \left| \frac{a^{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{\sigma(n+1)x^{n+1}}{(n+1)!}}{\frac{\sigma(n)x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{\sigma(n+1)x^{n+1}}{(n+1)!} \cdot \frac{n!}{\sigma(n)x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

since $|\sigma(n)| = 1$ for all n. It follows that the series may be freely rearranged without altering the sum for any value of x. We will regroup this series according to whether n is even or odd:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\sigma(n)x^n}{n!} &= \frac{x^0}{0!} + \frac{x^1}{1!} - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots \\ &= \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right) + \left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)\end{aligned}$$

Since $\dfrac{x^0}{0!} - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$ is the Taylor series at $a = 0$ of $\cos(x)$ and $\dfrac{x^1}{1!} - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$ is the Taylor series at $a = 0$ of $\sin(x)$, the given series is the Taylor series at $a = 0$ of $f(x) = \cos(x) + \sin(x)$. $\square$

f. *(Brute Force)* The Taylor series at a of $f(x)$ is the power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$ for $n \geq 1$ and $f^{(0)}(x) = f(x)$. We will grind out the first several derivatives of $f(x) = e^{2x}$ at $a = 0$ and look for a pattern we can plug into Taylor's formula:

n	0	1	2	3	4	$\cdots$
$f^{(n)}(x)$	e^{2x}	$2e^{2x}$	2^2e^{2x}	2^3e^{2x}	2^4e^{2x}	$\cdots$
$f^{(n)}(0)$	1	2	2^2	2^3	2^4	$\cdots$

It's pretty obvious that $f^{(n)}(0) = 2^n$ for all $n \geq 0$. (The paranoid may verify this with an argument by induction.) It now follows that the Taylor series at $a = 0$ of $f(x) = e^{2x}$ is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{2^n}{n!}x^n$. $\square$

f. *(Algebra)* The Taylor series at $a = 0$ of e^x is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. To get the Taylor series at $a = 0$ of e^{2x}, we simply plug in $2x$ for x in this series to get $\displaystyle\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!}x^n$. $\blacksquare$

4. Consider the region bounded by $y = 0$ and $y = \dfrac{1}{x}$ for $1 \leq x < \infty$.

a. Find the area of this region. *[4]*

b. Find the volume of the solid obtained by revolving the region about the x-axis. *[8]*

SOLUTIONS. **a.** Since $\dfrac{1}{x} > 0$ for all $x \geq 1$, the area of the region is:

$$\mathrm{A} = \int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln(x)\Big|_1^t = \lim_{t\to\infty} (\ln(t) - \ln(1)) = \infty - 0 = \infty \quad \square$$

b. We will use the disk method, so, since the region is revolved about the x-axis, we work in terms of x. The disk at x has radius $r = \dfrac{1}{x} - 0 = \dfrac{1}{x}$ and so has area $A(x) = \pi r^2 = \pi\left(\dfrac{1}{x}\right)^2 = \dfrac{\pi}{x^2}$. Thus the volume of the solid is:

$$\begin{aligned}\mathrm{V} &= \int_1^{\infty} A(x)\,dx = \int_1^{\infty} \frac{\pi}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{\pi}{x^2}\,dx = \lim_{t\to\infty} -\frac{\pi}{x}\Big|_1^t \\ &= \lim_{t\to\infty} \left[\left(-\frac{\pi}{t}\right) - \left(-\frac{\pi}{1}\right)\right] = \lim_{t\to\infty} \left[\frac{pi}{-}\frac{\pi}{t}\right] = \pi - 0 = \pi \quad \blacksquare\end{aligned}$$

Part B. Do either *one* (1) of **5** or **6**. *[14]*

5. Consider the piece of the parabola $y = \frac{1}{2}x^2$ for which $0 \le x \le 2$.

a. Find the arc-length of this piece. *[9]*

b. Find the area of the surface obtained by revolving this piece about the y-axis. *[5]*

SOLUTIONS. **a.** First, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}x^2\right) = \frac{1}{2} \cdot 2x = x$. We will use the trigonometric substitution $x = \tan(\theta)$, so $dx = \sec^2(\theta)\,d\theta$, as well as the reduction formula $\int \sec^3(\theta)\,d\theta = \frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\int \sec(\theta)\,d\theta$, to deal with the arc-length integral:

$$\begin{aligned}
\text{arc-length} &= \int_0^2 s = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 \sqrt{1+x^2}\,dx \\
&= \int_{x=0}^{x=2} \sqrt{1+\tan^2(\theta)}\,\sec^2(\theta)\,d\theta = \int_{x=0}^{x=2} \sec^3(\theta)\,d\theta \\
&= \frac{1}{2}\sec(\theta)\tan(\theta)\Big|_{x=0}^{x=2} + \frac{1}{2}\int_{x=0}^{x=2} \sec(\theta)\,d\theta \\
&= \frac{1}{2}x\sqrt{1+x^2}\Big|_0^2 + \frac{1}{2}\ln\left(\tan(\theta)+\sec(\theta)\right)\Big|_{x=0}^{x=2} \\
&= \frac{1}{2}\cdot 2\sqrt{5} - \frac{1}{2}\cdot 0\sqrt{1} + \ln\left(x + \sqrt{1+x^2}\right)\Big|_0^2 \\
&= \sqrt{5} + \ln\left(2+\sqrt{5}\right) - \ln\left(0+\sqrt{1}\right) = \sqrt{5} + \ln\left(2+\sqrt{5}\right) \quad \square
\end{aligned}$$

b. As above, we have $\frac{dy}{dx} = x$. Since we are revolving the curve about the y-axis, the point on the curve at x gets revolved through a circle of radius $r = x - 0 = x$. We will use the substitution $u = 1 + x^2$, so $du = 2x\,dx$ and $\begin{matrix} x & 0 & 2 \\ u & 1 & 5 \end{matrix}$, to deal with the resulting integral for the area of the surface of revolution.

$$\begin{aligned}
\text{SA} &= \int_0^2 2\pi r\,ds = \int_0^2 2\pi x\sqrt{1+x^2}\,dx = \int_1^5 \pi\sqrt{u}\,du = \int_1^5 \pi u^{1/2}\,du = \frac{2}{3}\pi u^{3/2}\Big|_1^5 \\
&= \frac{2}{3}\pi\cdot 5^{3/2} - \frac{2}{3}\pi\cdot 1^{3/2} = \frac{10\sqrt{5}}{3}\pi - \frac{2}{3}\pi = \frac{10\sqrt{5}-2}{3}\pi \quad \blacksquare
\end{aligned}$$

6. The region below $y = -x^2 + 4x - 3$ and above $y = 0$ for $1 \le x \le 3$ is revolved about the line $x = -1$. Find the volume of the resulting solid. *[14]*

SOLUTION. We will use the method of cylindrical shells to find the volume. (One could use the disk/washer method, but there would be substantial overhead in terms of algebraic complexity in this case.) Since we are revolving about a vertical line and using shells, we will work in terms of x. The shell at x has radius $r = x - (-1) = x + 1$ and height $h = y - 0 = -x^2 + 4x - 3$, and hence area $A(x) = 2\pi rh = 2\pi(x+1)\left(-x^2 + 4x - 3\right) = 2\pi\left(-x^3 + 3x^2 + x - 3\right)$. The volume of the solid of revolution is then given by:

$$\begin{aligned}
\mathrm{V} &= \int_1^3 A(x)\,dx = 2\pi \int_1^3 \left(-x^3 + 3x^2 + x - 3\right)\,dx = 2\pi\left(-\frac{1}{4}x^4 + x^3 + \frac{1}{2}x^2 - 3x\right)\Bigg|_1^3 \\
&= 2\pi\left(-\frac{1}{4}\cdot 3^4 + 3^3 + \frac{1}{2}\cdot 3^2 - 3\cdot 3\right) - 2\pi\left(-\frac{1}{4}\cdot 0^4 + 0^3 + \frac{1}{2}\cdot 0^2 - 3\cdot 0\right) \\
&= 2\pi\left(-\frac{81}{4} + 27 + \frac{9}{2} - 9\right) - 2\pi\cdot 0 = 2\pi\cdot\frac{9}{4} = \frac{9}{2}\pi \quad \blacksquare
\end{aligned}$$

Part C. Do either *one* (1) of **7** or **8**. *[14]*

7. Find the Taylor series at $a = 0$ of $f(x) = \dfrac{2}{x+2}$ and determine its radius and interval of convergence.

SOLUTION. *(Brute Force)* The Taylor series at a of $f(x)$ is the power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$ for $n \geq 1$ and $f^{(0)}(x) = f(x)$. We will grind out the first several derivatives of $f(x) = \dfrac{2}{x+2}$ at $a = 0$ and look for a pattern we can plug into Taylor's formula:

n	0	1	2	3	4	$\cdots$
$f^{(n)}(x)$	$\frac{2}{x+2}$	$-\frac{2}{(x+2)^2}$	$\frac{4}{(x+2)^3}$	$-\frac{12}{(x+2)^4}$	$\cdots$	
$f^{(n)}(0)$	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{4}$	$\cdots$	

A little reflection about what's going on in the second line of the table tells us that $f^{(n)}(x) = \dfrac{2\cdot(-1)^n\cdot n!}{(x+2)^{n+1}}$. It follows that $f^{(n)}(0) = \dfrac{2\cdot(-1)^n\cdot n!}{(0+2)^{n+1}} = \dfrac{(-1)^n n!}{2^n}$, and so the Taylor series at $a = 0$ of $f(x) = \dfrac{2}{x+2}$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n}\cdot\frac{(x-0)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}x^n$$

It remains to determine the radius and interval of convergence of this series. Just for kicks, we'll use the Root Test, though the Ratio Test works equally well here:

$$\lim_{n\to infty} \left|\frac{(-1)^n}{2^n}x^n\right|^{1/n} = \lim_{n\to infty} \left(\left|\frac{x}{2}\right|^n\right)^{1/n} = \lim_{n\to infty} \left|\frac{x}{2}\right| = \frac{|x|}{2}$$

Since $\dfrac{|x|}{2} < 1$ exactly when $|x| < 2$, the Root tells us tells us that radius of convergence of the Taylor series is 2.

To finish sorting out the interval of convergence, we check what happens at the endpoints, $x = -2$ and $x = 2$. When $x = -2$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n 2^n}{2^n} \sum_{n=0}^{\infty}(-1)^{2n} = \sum_{n=0}^{\infty} 1$, which diverges by the Divergence Test because $\displaystyle\lim_{n\to\infty} 1 = 1 \neq 0$. Similarly, when $x = 2$, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} 2^n = \sum_{n=0}^{\infty} (-1)^n$, which diverges by the Divergence Test because $\displaystyle\lim_{n\to\infty} (-1)^n$ fails to exist, much less equal 0. It follows that the interval of convergence of the Taylor series is $(-2, 2)$. □

Solution. *(Algebra)* $f(x) = \dfrac{2}{x+2}$ looks somewhat similar to the formula for the sum of a geometric series, which is $\frac{a}{1-r}$ for the geometric series that has first term a and common ratio r (where we ned to have $|r| < 1$ for this to work). We will do a bit of algebra to the defining formula for $f(x)$ to put it in the form of a sum for a geometric series:

$$f(x) = \frac{2}{2+x} = \frac{2}{2+x} \cdot \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{\frac{2}{2}}{\frac{2}{2} + \frac{x}{2}} = \frac{1}{1 + \frac{x}{2}} = \frac{1}{1 - \left(-\frac{x}{2}\right)}$$

Thus $f(x)$ is the sum of a geometric series with first term 1 and common ratio $-\dfrac{x}{2}$, so $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}$, which must be the Taylor series of the function since every power series is its own Taylor series. Note that this geometric series converges exactly when the common ratio satisfies $\left|-\dfrac{x}{2}\right| = \dfrac{|x|}{2} < 1$, *i.e.* exactly when $|x| < 2$. It follows that the radius of convergence of the series is 2 and the interval of convergence is $(-2, 2)$. ■

8. Find the Taylor series at $a = 1$ of $f(x) = \dfrac{2}{1+x}$ and determine its radius and interval of convergence.

Solution. Either solution to **7** can be executed here with only minor changes because $f(x) = \dfrac{2}{1+x} = \dfrac{2}{2 + (x-1)}$, which is the same function that we have in **7**, except with $x - 1$ plugged in for x. The radius of convergence is also 2, and the interval of convergence is $(-2+1, 2+1) = (-1, 3)$ (*i.e.* the open interval of width 2 centered at 1 instead of 0). ■

[Total = 100]

Part D. Bonus problems! If you feel like it and have the time, do one or both of these.

Δ. What does the infinite product $\displaystyle 2\prod_{n=1}^{\infty} \left[\frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\right] = 2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \cdots$ amount to? *[1]*

Solution. This product, discovered by John Wallis (1616-1703), equals π. Takes a bit of work to prove that mind you ... :-) ▲

□. Write a haiku (or several :-) touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

Solution. You're on your own! ■

Enjoy the rest of your summer!