Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, SUMMER 2011

Solutions to Assignment #7
An integral inequality

$\genfrac{}{}{0pt}{}{\text{Up}}{\text{Down}}$side: No `Maple`; it won't help. $\genfrac{}{}{0pt}{}{\text{Down}}{\text{Up}}$side: It's a proof. (Well, a generic calculation or two, anyway.)

1. Suppose that $f(x)$ and $g(x)$ are continuous functions which are not always equal to 0 on some interval $[a, b]$. Show that

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right). \text{ [10]}$$

NOTE: To do this you will probably want to review some of the basic properties of definite integrals, especially the order properties, given in Chapter 5 of the textbook.

Hint: Consider the case where there is some constant c such that $f(x) = cg(x)$ for all x in $[a, b]$ separately from the case where there is no such constant.

SOLUTION. Following the hint, we will consider the two cases separately, easier first.

Case 1: There is some constant c such that $f(x) = cg(x)$ for all x in $[a, b]$.

We actually get equality in this case:

$$\begin{aligned}\left(\int_a^b f(x)g(x)\,dx\right)^2 &= \left(\int_a^b cg(x)g(x)\,dx\right)^2 = \left(c\int_a^b g^2(x)\,dx\right)^2 \\ &= c^2\left(\int_a^b g^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right) \\ &= \left(\int_a^b c^2g^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right) \\ &= \left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right)\end{aligned}$$

Case 2: For any constant c, there is some $x \in [a, b]$ for which $f(x) \neq cg(x)$.

Suppose c is any constant. Note that since $f(x)$ and $g(x)$ are continuous on $[a, b]$, if $f(x) \neq cg(x)$ for some x, then there is a whole subinterval of $[a, b]$ on which $f(x) \neq cg(x)$. It follows that $(f(x) - cg(x))^2 > 0$ on some subinterval of $[a, b]$. This in turn means that

$$\begin{aligned}0 &< \int_a^b (f(x) - cg(x))^2\,dx = \int_a^b \left(f^2(x) - 2cf(x)g(x) + c^2g^2(x)\right)dx \\ &= \int_a^b f^2(x)\,dx - 2c\int_a^b f(x)g(x)\,dx + c^2\int_a^b g^2(x)\,dx\,;\end{aligned}$$

that is

$$2c \int_a^b f(x)g(x)\,dx < \int_a^b f^2(x)\,dx + c^2 \int_a^b g^2(x)\,dx$$

for *any* constant c.

In particular, this last inequality must hold if

$$c = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g^2(x)\,dx}\,.$$

Plugging this in gives

$$2\frac{\left(\int_a^b f(x)g(x)\,dx\right)^2}{\int_a^b g^2(x)\,dx} < \int_a^b f^2(x)\,dx + \frac{\left(\int_a^b f(x)g(x)\,dx\right)^2}{\int_a^b g^2(x)\,dx}\,,$$

which simplifies to

$$\frac{\left(\int_a^b f(x)g(x)\,dx\right)^2}{\int_a^b g^2(x)\,dx} < \int_a^b f^2(x)\,dx\,,$$

and multiplying through by $\int_a^b g^2(x)\,dx$ now gives

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 < \left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right)\,,$$

as desired.

Please note that this argument fails if $\int_a^b g^2(x)\,dx = 0$. (Dividing by 0 is *bad!*) This will not occur here, though, because $g(x)$ was assumed to be continuous and not always equal to 0 on $[a,b]$, from which it follows that $g^2(x) > 0$ on (at least) some subinterval of $[a,b]$. This, in turn, implies that $\int_a^b g^2(x)\,dx > 0$.

It turns out that the result still works if we drop the requirement that $g(x)$ is not always 0 on $[a,b]$, allowing $\int_a^b g^2(x)\,dx = 0$. Why does it? $\Box$

Bonus: A two-player game (in which the players take turns making moves) is considered to be finite if it cannot go on forever when played by the rules. The two-player game SUPERGAME is played as follows: the first player chooses a finite two-player game, which the two players proceed to play out with the second player going first. Is SUPERGAME itself a finite two-player game? Why or why not? *[1]*

SOLUTION. SUPERGAME cannot itself be a finite game because a contradiction would arise if it were:

> Suppose SUPERGAME were a finite game. Then choosing SUPERGAME would be a valid first move in a game of SUPERGAME. But then a legal game of SUPERGAME could go on forever: the first player makes the first move and chooses SUPERGAME as the finite game to be played out. Now the second player has to make the first move in a game of SUPERGAME, and chooses SUPERGAME too. This leaves the first player to make the first move in a game of SUPERGAME, who chooses SUPERGAME again, which leaves the second player to make the the first move in a game of SUPERGAME, who ...

The fundamental problem here is that the notion of "finite two-player game", in terms of which SUPERGAME was defined, was not really precise. (Heck, the notion of "game" was never precisely defined here. How *would* you do that?) To avoid the contradiction above, any attempt to make the notion of "finite two-player game" really precise would have to have to be sufficiently restrictive to exclude SUPERGAME. □