Mathematics 1110H (Section A) – Calculus I: Limits, Derivatives, and Integrals
Trent University, Fall 2024

Solutions to Quiz #5
Max & Min ran down the hill

Please show all your work. Don't be shy about using a calculator or a computer.

1. Find the maximum and minimum values of $g(x) = x^3 - 9x^2 + 23x - 15$ on the interval $[0.5, 5.5]$. *[5]*

Solution. $g(x)$ is a polynomial, so it is defined and continuous for all x, and $[0.5, 5.5]$ is a closed interval, so we can find the maximum and minimum values of $g(x)$ on the interval by finding and comparing its values at the endpoints of and any critical points in the interval.

Endpoints. No calculus, but too much arithmetic ...

$$\begin{aligned}
g(0.5) &= g\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 9\left(\frac{1}{2}\right)^2 + 23 \cdot \frac{1}{2} - 15 = \frac{1}{8} - \frac{9}{4} + \frac{23}{2} - 15 \\
&= \frac{1}{8} - \frac{18}{8} + \frac{92}{8} - \frac{120}{8} = \frac{1 - 18 + 92 - 120}{8} = -\frac{45}{8} = -5.625 \\
g(5.5) &= g\left(\frac{11}{2}\right) = \left(\frac{11}{2}\right)^3 - 9\left(\frac{11}{2}\right)^2 + 23 \cdot \frac{11}{2} - 15 \\
&= \frac{1331}{8} - \frac{1089}{4} + \frac{253}{2} - 15 = \frac{1331}{8} - \frac{2178}{8} + \frac{1012}{8} - \frac{120}{8} \\
&= \frac{1331 - 2178 + 1012 - 120}{8} = \frac{45}{8} = 5.625
\end{aligned}$$

OK, doing all the arithmetic by hand was a real drag. Let's have SageMath do it, just to be sure:

[1]:
```
g = function('g')(x)
g(x) = x^3 - 9*x^2 + 23*x - 15
g(0.5)
```

[1]: -5.62500000000000

[2]:
```
g(5.5)
```

[2]: 5.62500000000000

It seems the arithmetic was done correctly!

Critical points. Calculus, since we need to find the points where $g'(x) = \frac{d}{dx}g(x) = 0$ and evaluate $g(x)$ at those points. Sadly, this has even more arithmetic ...

$$g'(x) = \frac{d}{dx}\left(x^3 - 9x^2 + 23x - 15\right) = 3x^2 - 18x + 23$$

We find the x such that $g'(x) = 3x^2 - 18x + 23 = 0$ using the quadratic formula:

$$\begin{aligned} x &= \frac{-(-18) \pm \sqrt{(-18)^2 - 4 \cdot 3 \cdot 23}}{2 \cdot 3} = \frac{18 \pm \sqrt{324 - 278}}{6} \\ &= \frac{18 \pm \sqrt{48}}{6} = \frac{18 \pm 4\sqrt{3}}{6} = \frac{9 \pm 2\sqrt{3}}{3} = 3 \pm \frac{2}{\sqrt{3}} \approx 3 \pm 1.1547 \end{aligned}$$

Let's check this with SageMath:

```
[3]: solve( diff(g(x),x) == 0, x )
```

```
[3]: [x == -2/3*sqrt(3) + 3, x == 2/3*sqrt(3) + 3]
```

It agrees, since $\frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$. Thus $g'(x) = 0$ when $x = 3 + \frac{2}{\sqrt{3}} \approx 4.1547$ and when $x = 3 - \frac{2}{\sqrt{3}} \approx 1.8453$, both of which are in the interval $[0.5, 5.5]$. We evaluate $g(x)$ at these points using SageMath – doing it by hand would be too awful to bear:

```
[4]: g(-2/3*sqrt(3) + 3)
```

```
[4]: -1/27*(2*sqrt(3) - 9)^3 - (2*sqrt(3) - 9)^2 - 46/3*sqrt(3) + 54
```

```
[5]: N(-1/27*(2*sqrt(3) - 9)^3 - (2*sqrt(3) - 9)^2 - 46/3*sqrt(3) + 54)
```

```
[5]: 3.07920143567800
```

```
[6]: g(2/3*sqrt(3) + 3)
```

```
[6]: 1/27*(2*sqrt(3) + 9)^3 - (2*sqrt(3) + 9)^2 + 46/3*sqrt(3) + 54
```

```
[7]: N(1/27*(2*sqrt(3) + 9)^3 - (2*sqrt(3) + 9)^2 + 46/3*sqrt(3) + 54)
```

```
[7]: -3.07920143567800
```

That is, $g\left(3 + \frac{2}{\sqrt{3}}\right) \approx 3.0792$ and $g\left(3 - \frac{2}{\sqrt{3}}\right) \approx -3.0792$.

Conclusion. Comparing the values of $g(x)$ at the endpoints of the interval and the critical points in the interval, we conclude that the maximum value of $g(x)$ on the interval $[0.5, 5.5]$, namely 5.625, occurs at the right-hand endpoint of the interval, $x = 5.5$, and the minimum value, namely -5.625, occurs at the left-hand endpoint of the interval, $x = 0.5$. ■