

Mathematics 1101Y – Calculus I: Functions and calculus of one variable

TRENT UNIVERSITY, 2013–2014

Solutions to Assignment #2

Solving equations with and without Maple

Before attempting the questions below, take another peek at Section 1.6 of the text for the definitions of the various hyperbolic functions. It might also be useful to skim through *Getting started with Maple 10* by Gilberto E. Urroz – read those parts concerned with solving equations more closely. (You can find a link to it on the MATH 1101Y web page.) Maple’s help facility may also come in handy, especially when trying to make out the intricacies of what the `solve` command [*Hint! Hint!*] and its options and variations do. Make use of the Maple labs, too!

We won’t see much of the hyperbolic trig functions in this course, but they do provide various examples of things that are just too juicy to pass up entirely, such as our example in class of finding the inverse function of $\cosh(x)$. In this assignment we will do the same for $\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$, among other things.

1. Explain why every point on the parametric curve $x = \operatorname{sech}(t)$ and $y = \tanh(t)$, $-\infty < t < \infty$, is on the circle $x^2 + y^2 = 1$. Do you get every point on the circle this way? Explain why or why not. [2]

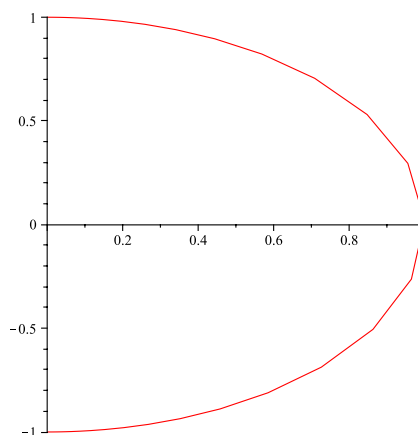
SOLUTION. Suppose $x = \operatorname{sech}(t)$ and $y = \tanh(t)$ for any t with $-\infty < t < \infty$. Then, with the help of the hyperbolic identity $\cosh^2(x) = 1 + \sinh^2(x)$,

$$x^2 + y^2 = \operatorname{sech}^2(x) + \tanh^2(x) = \frac{1}{\cosh^2(x)} + \frac{\sinh^2(x)}{\cosh^2(x)} = \frac{1 + \sinh^2(x)}{\cosh^2(x)} = \frac{\cosh^2(x)}{\cosh^2(x)} = 1,$$

so the point $(x, y) = (\operatorname{sech}(t), \tanh(t))$ is on the circle $x^2 + y^2 = 1$.

On the other hand, since $1 \leq \cosh(t) < \infty$ for $-\infty < t < \infty$, we have $0 < x = \operatorname{sech}(t) = \frac{1}{\cosh(t)} \leq 1$ for $-\infty < t < \infty$, so the given parametric curve does not include any points for which $x \leq 0$. This means it consists of at most the right half of the circle $x^2 + y^2 = 1$. A quick look at the graph,

```
> plot([sech(t), tanh(t), t=-1000..1000])
```

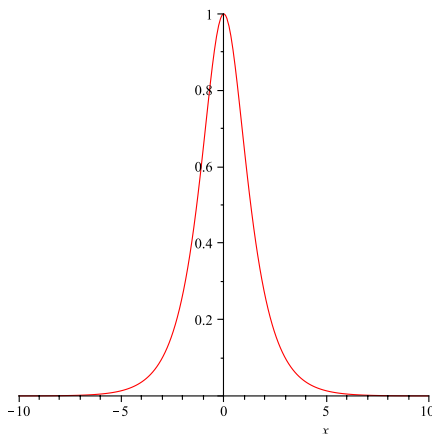


confirms that that is exactly what it consists of. ■

2. What are the domain and range of $\operatorname{sech}(x)$? Use **Maple** to plot the curve $y = \operatorname{sech}(x)$. [Please submit a printout of your worksheet.] On the basis of your plot, how much of $\operatorname{sech}(x)$ should be invertible? [1]

SOLUTION. $\cosh(x) = \frac{e^x + e^{-x}}{2}$ is defined and > 0 for all x ; it follows that $\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$ is also defined and > 0 for all x . As noted above in the solution to question 1, since $1 \leq \cosh(x) < \infty$ for all x , $0 < \operatorname{sech}(x) = \frac{1}{\cosh(x)} \leq 1$. This can be confirmed by taking a look at the graph of $\operatorname{sech}(x)$:

`> plot(sech(x))`



Thus the domain of $\operatorname{sech}(x)$ is $\mathbb{R} = (-\infty, \infty)$ and its range is $(0, 1]$. ■

3. Find an expression – by hand! – for $\operatorname{arcsech}(x)$, the inverse function of $\operatorname{sech}(x)$, in terms of the natural logarithm function. [2]

SOLUTION. This is very similar to finding an expression for $\operatorname{arccosh}(x)$, the inverse of $\cosh(x)$, which we did in class some weeks ago:

$$\begin{aligned} y = \operatorname{arcsech}(x) &\iff x = \operatorname{sech}(y) = \frac{1}{\cosh(y)} = \frac{2}{e^y + e^{-y}} \\ &\iff x(e^y + e^{-y}) = 2 \\ &\iff xe^y + \frac{x}{e^y} = 2 \\ &\iff xe^y e^y + \frac{x}{e^y} e^y = 2e^y \\ &\iff x(e^y)^2 + x = 2e^y \\ &\iff x(e^y)^2 - 2e^y + x = 0 \end{aligned}$$

This last is a quadratic equation in e^y , so we use the quadratic formula to solve for e^y before we try to solve for y :

$$\begin{aligned} x(e^y)^2 - 2e^y + x = 0 &\iff e^y = \frac{-(-2) \pm \sqrt{(-2)^2 - 4xx}}{2x} = \frac{2 \pm \sqrt{4 - 4x^2}}{2x} \\ &= \frac{2 \pm \sqrt{4(1 - x^2)}}{2x} = \frac{2 \pm 2\sqrt{1 - x^2}}{2x} = \frac{1 \pm \sqrt{1 - x^2}}{x} \\ &= \frac{1}{x} \pm \frac{1}{x}\sqrt{1 - x^2} \end{aligned}$$

Thus $y = \ln(e^y) = \ln\left(\frac{1}{x} \pm \frac{1}{x}\sqrt{1 - x^2}\right)$. Note that this is really two possible expressions, namely $y = \ln\left(\frac{1}{x} + \frac{1}{x}\sqrt{1 - x^2}\right)$ and $y = \ln\left(\frac{1}{x} - \frac{1}{x}\sqrt{1 - x^2}\right)$, respectively. ■

4. Use **Maple** to find an expression for $\operatorname{arcsech}(x)$ in terms of the natural logarithm function. [Please submit a printout of your worksheet.] Is this the same as the one you obtained by hand? [2]

SOLUTION. We use the `solve` command in worksheet mode, making sure to use `exp(y)` instead of `e^y` for the function e^y to help avoid the problem that in worksheet mode **Maple** interprets `e` as an unspecified constant instead of the familiar constant e .

> solve(2/(exp(y)+exp(-y))=x,y)

$$\ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right), \ln\left(\frac{1 - \sqrt{1 - x^2}}{x}\right)$$

Note that this is basically the same expression we obtained in the solution to **3** above. ■

5. What are the domain and range of $\operatorname{arcsech}(x)$? [Your version of the expression, if it's different from **Maple**'s.] [2]

SOLUTION. To determine the domain, the basic constraints here are that \sqrt{t} is defined only for $t \geq 0$, that $\ln(t)$ is only defined for $t > 0$, and that we cannot divide by 0.

It follows from the first constraint that both of the expressions we obtained for $\operatorname{arcsech}(x)$, $\ln\left(\frac{1}{x} + \frac{1}{x}\sqrt{1 - x^2}\right)$ and $\ln\left(\frac{1}{x} - \frac{1}{x}\sqrt{1 - x^2}\right)$, will require that $1 - x^2 \geq 0$, that is, that $x^2 \leq 1$, which is true exactly when $-1 \leq x \leq 1$.

The last constraint, not dividing by 0, forces us to limit the possible inputs x to $-1 \leq x < 0$ or $0 < x \leq 1$.

The second constraint, that the input to $\ln(t)$ be positive, along with the fact that $1 + \sqrt{1 - x^2} \geq 1 > 0$ whenever it is defined, means that $\ln\left(\frac{1}{x} + \frac{1}{x}\sqrt{1 - x^2}\right) = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right)$ is not defined for negative x , so its domain is $0 < x \leq 1$. Similarly, since $1 - \sqrt{1 - x^2} > 0$ (as $\sqrt{1 - x^2} < 1$ when $\sqrt{1 - x^2}$ is defined and $x > 0$), $\ln\left(\frac{1}{x} - \frac{1}{x}\sqrt{1 - x^2}\right) = \ln\left(\frac{1 - \sqrt{1 - x^2}}{x}\right)$ is also undefined for negative x , and so its domain is also $0 < x \leq 1$.

To determine the range, recall first that $\ln(t)$ is an increasing function with range $(-\infty, \infty)$ on the domain $(0, \infty)$. In particular, for $0 < t \leq 1$, $\ln(t)$ has range $(-\infty, 0]$, and for $1 \leq t < \infty$, $\ln(t)$ has range $[0, \infty)$.

Second, for $0 < x \leq 1$ we get $1 \leq \frac{1+\sqrt{1-x^2}}{x} < \infty$ – when $x = 1$, $\frac{1+\sqrt{1-1^2}}{1} = \frac{1+0}{1} = 1$, and when $x \rightarrow 0^+$, $\frac{1+\sqrt{1-x^2}}{x} \geq \frac{1}{x} \rightarrow \infty$. It follows that $\ln\left(\frac{1}{x} + \frac{1}{x}\sqrt{1-x^2}\right) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ has range $[1, \infty)$ for x in its domain of $(0, 1]$.

Third, for $0 < x \leq 1$ we get $0 < \frac{1-\sqrt{1-x^2}}{x} \leq 1$ – when $x = 1$, $\frac{1-\sqrt{1-1^2}}{1} = \frac{1-0}{1} = 1$, and when $x \rightarrow 0^+$, $\frac{1-\sqrt{1-x^2}}{x} \rightarrow 0^+$. To see the latter, note that $\frac{1-\sqrt{1-x^2}}{x} = \frac{1-\sqrt{1-x^2}}{x} \cdot \frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \frac{1-1+x^2}{x(1+\sqrt{1-x^2})} = \frac{x}{1+\sqrt{1-x^2}}$. Since, as noted in the previous paragraph, $\frac{1+\sqrt{1-x^2}}{x} \rightarrow \infty$ as $x \rightarrow 0^+$, it follows that $\frac{1-\sqrt{1-x^2}}{x} = \frac{x}{1+\sqrt{1-x^2}} \rightarrow 0^+$ as $x \rightarrow 0^+$. Thus $\ln\left(\frac{1}{x} - \frac{1}{x}\sqrt{1-x^2}\right) = \ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)$ has range $(-\infty, 1]$ for x in its domain of $(0, 1]$.

What is going on here is that $\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ inverts $\operatorname{sech}(x)$ for $x \geq 0$, while $\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)$ inverts $\operatorname{sech}(x)$ for $x \leq 0$. ■

- 6.** Just for fun – and a mark too! – use **Maple** to find any and all the real roots of the cubic equation, $x^3 - 2x^2 + 3x - 4 = 0$, to 10 decimal places. [1]

SOLUTION. **fsolve** is variant of the **solve** command that tries to find numeric solutions.

```
> fsolve(x^3-2*x^2+3*x-4=0,x)
```

1.650629191

This would have received full-credit, but the sharp-eyed may notice that while this has 10 significant digits, one of those is before the decimal point. You can tell **Maple** to use more significant digits in floating point calculations, and thus get more after the decimal point. For example,

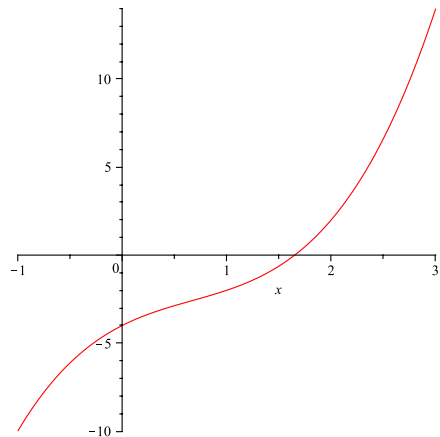
```
> Digits:=15
> fsolve(x^3-2*x^2+3*x-4=0,x,fulldigits)
```

1.65062919143939

gives us 15 significant digits, with 14 after the decimal point. You figure out how to get exactly the answer asked for!

The slightly sceptical may wonder whether there are any other solutions, since, after all, a cubic may have as many three roots. **solve** and its variants do try to find all possible solutions, but may occasionally fail to do so. Plotting the curve

```
> plot(x^3-2*x^2+3*x-4,x=-1..3)
```



does suggest that there is only one solution in this case, however. ■

REFERENCES

1. *Calculus: Early Transcendentals* (2nd Edition), by Jon Rogawski, W.H. Freeman, New York, 2012, ISBN-10: 1-4292-6009-2, ISBN-13: 978-1-4292-6009-1.
2. *Getting started with Maple 10*, by Gilberto E. Urroz (2005), which can found (pdf) at: <http://euclid.trentu.ca/math/sb/1101Y/2012-2013/GettingStartedMaple10.pdf>