Mathematics 1101Y – Calculus I: Functions and calculus of one variable
TRENT UNIVERSITY, 2012–2013

Solutions to Assignment #4
Breaking limits

One of the things we've skipped over was the formal definition of limit, that is, how to pin down just what $\lim_{x \to a} f(x) = L$ really means. The usual definition of limits is something like:

> $\varepsilon - \delta$ DEFINITION OF LIMITS. $\lim_{x \to a} f(x) = L$ exactly when for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any x with $|x - a| < \delta$ we are guaranteed to have $|f(x) - L| < \varepsilon$ as well.

Informally, this means that no matter how close – that's the ε – you want $f(x)$ to get to L, you can make it happen by ensuring that x is close enough – that's the δ – to a. If this can always be done, $\lim_{x \to a} f(x) = L$; if not, then $\lim_{x \to a} f(x) \neq L$.

This definition works, but most people find it a little hard to understand and use at first. Here is less common definition equivalent to the one above that is cast in terms of a game:

> LIMIT GAME DEFINITION OF LIMITS. The *limit game* for $f(x)$ at $x = a$ with target L is a three-move game played between two players A and B as follows:
>
> 1. A moves first, picking a small number $\varepsilon > 0$.
> 2. B moves second, picking another small number $\delta > 0$.
> 3. A moves third, picking an x that is within δ of a, *i.e.* $a - \delta < x < a + \delta$.
>
> To determine the winner, we evaluate $f(x)$. If it is within ε of the target L, *i.e.* $L - \varepsilon < f(x) < L + \varepsilon$, then player B wins; if not, then player A wins.
>
> With this idea in hand $\lim_{x \to a} f(x) = L$ means that player B has a winning strategy in the limit game for $f(x)$ at $x = a$ with target L; that is, if B plays it right, B will win no matter what A tries to do. (At least within the rules ... :-) Conversely, $\lim_{x \to a} f(x) \neq L$ means that player A is the one with a winning strategy in the limit game for $f(x)$ at $x = a$ with target L.

Your task in this assignment, should you choose to accept it, is to find such winning strategies:

1. Describe a winning strategy for B in the limit game for $f(x) = 3x - 2$ at $x = 2$ with target 4. Note that no matter what number ε A plays first, B must have a way to find a δ to play that will make it impossible for A to play an x that wins for A on the third move. *[3]*

SOLUTION. Whatever $\varepsilon > 0$ A may play, B will win by responding with $\delta = \dfrac{\varepsilon}{3}$. (Any positive δ which is even smaller will also work.) No matter what x A chooses with $|x - 2| < \delta = \dfrac{\varepsilon}{3}$, we have

$$|f(x) - 4| = |(3x - 2) - 4| = |3x - 6| = 3|x - 2| < 3\delta = 3\frac{\varepsilon}{3} = \varepsilon\,,$$

so B wins. ■

NOTE: How does one get $\delta = \varepsilon/3$? By reverse-engineering the δ from the desired conclusion, $|f(x) - 4| < \varepsilon$:

$$|f(x) - 4| < \varepsilon \;\Leftrightarrow\; |(3x - 2) - 4| < \varepsilon \;\Leftrightarrow\; |3x - 6| < \varepsilon \;\Leftrightarrow\; 3|x - 2| < \varepsilon \;\Leftrightarrow\; |x - 2| < \frac{\varepsilon}{3} \qquad ■$$

2. Describe a winning strategy for A in the limit game for $f(x) = 3x - 2$ at $x = 2$ with target 5. Note that A must pick an ε on the first move so that no matter what δ B tries to play on the second move, A can still find an x to play on move three that wins for A. *[3]*

SOLUTION. For the first move, let A play $\varepsilon = \dfrac{1}{2}$. (Any positive $\varepsilon \le 1$ will also work.) No matter what $\delta > 0$ B plays in response, A can respond in turn with any x such that $2 - \delta < x < 2$. Since

$$x < 2 \implies f(x) = 3x - 2 < 3 \cdot 2 - 2 = 4 < 4.5 = 5 - \frac{1}{2} = 5 - \varepsilon\,,$$

so $|f(x) - 5| \ge \frac{1}{2} = \varepsilon$, which means that A wins. ■

NOTE: How does one figure out what ε to pick to begin with? You need one that is small enough to separate the target, 5, from where the function is really going, $f(2) = 4$. That is, any ε that is less than the distance between these two numbers, $5 - 4 = 1$, will do. (Now, why does $\varepsilon = 1$ still – barely – do the job?)

3. Use either definition of limits above to verify that $\lim_{x \to 1} \left(x^2 + x + 1\right) = 3$. *[4]*

Hint: The choice of δ will probably require some slightly indirect reasoning. Pick some arbitrary smallish positive number, say 0.5, for δ as a first cut. If it doesn't do the job, but x is at least that close, you'll have some more information to help pin down the δ you really need.

SOLUTION. As in the note after the solution to problem **1**, we will attempt to reverse-engineer the $\delta > 0$ required:

$$\begin{aligned} |f(x) - L| < \varepsilon \;\Leftrightarrow\; \left|\left(x^2 + x + 1\right) - 3\right| < \varepsilon \;&\Leftrightarrow\; \left|x^2 + x - 2\right| < \varepsilon \\ \Leftrightarrow\; |(x+2)(x-1)| < \varepsilon \;&\Leftrightarrow\; |x - 1| < \frac{\varepsilon}{|x+2|} \end{aligned}$$

The problem is that δ may not depend on x. (Recall that A plays x *after* B plays δ.) Following the hint, we get around this problem by accepting no δ greater than 0.5. (Any number that is less than the distance between $x = 1$ and $x = -2$ will do, actually.) This lets us put bounds around $|x + 2|$ and so replace it with a constant in $\frac{\varepsilon}{|x+2|}$ above. If $|x - 1| < \delta \le 0.5$, then

$$\begin{aligned} -0.5 < x - 1 < 0.5 \;&\Leftrightarrow\; 0.5 = -0.5 + 1 < x = x - 1 + 1 < 0.5 + 1 = 1.5 \\ &\Leftrightarrow\; 2.5 = 0.5 + 2 < x + 2 < 1.5 + 2 = 3.5 \\ &\Leftrightarrow\; \frac{2}{5} = \frac{1}{2.5} > \frac{1}{x+2} > \frac{1}{3.5} = \frac{2}{7}\,. \end{aligned}$$

Note that it also follows that if $|x - 1| < \delta \le 0.5$, then $x + 2 > 0$ and so $|x + 2| = x + 2$. Thus, if $|x - 1| < \delta \le 0.5$, we have $|x - 1| < \frac{2\varepsilon}{5} < \frac{\varepsilon}{|x+2|}$.

It follows that if we let $\delta = \min\left(0.5, \frac{2\varepsilon}{5}\right)$, the lesser of 0.5 and $\frac{2\varepsilon}{3}$, then no matter what x is chosen that satisfies $|x - 1| < \delta$, we get that $\delta \le 0.5$, and so

$$\begin{aligned} |x - 1| < \min\left(0.5, \frac{2\varepsilon}{5}\right) \le \frac{2\varepsilon}{5} < \frac{\varepsilon}{|x+2|} \;&\Rightarrow\; |(x+2)(x-1)| < \varepsilon \\ &\Rightarrow\; \left|x^2 + x - 2\right| < \varepsilon \\ &\Rightarrow\; \left|\left(x^2 + x + 1\right) - 3\right| < \varepsilon\,, \end{aligned}$$

as required to show that $\lim_{x \to 1} \left(x^2 + x + 1\right) = 3$. ■