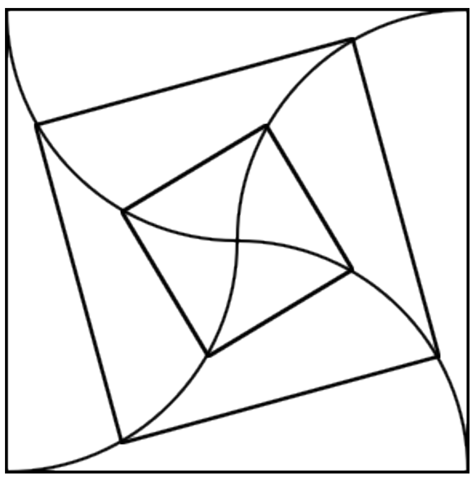
Mathematics 1101Y – Calculus I: Functions and calculus of one variable
Trent University, 2012–2013

Solutions to Assignment #3
The Beetles

Four beetles are placed on the corners of a square with sides of length 10 *cm*. Simultaneously, each beetle begins to crawl straight towards the next beetle in the next (considered counterclockwise) corner. Each beetle crawls at the same constant rate as every other beetle and at every instant it crawls directly towards its target beetle, so the locations of the beetles form a square at every instant. Eventually, of course, they all meet at the centre of the starting square.



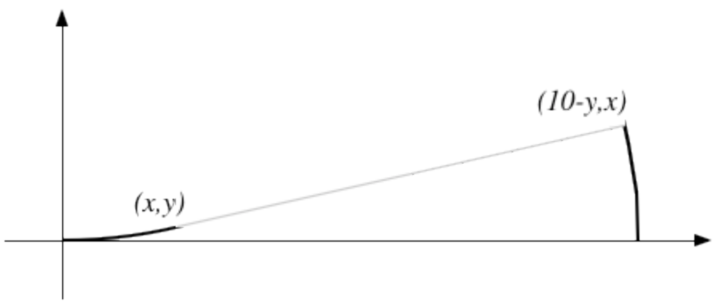
For questions **1**–**3**, suppose the beetles start out at $(0,0)$, $(10,0)$, $(10,10)$, and $(0,10)$. The beetle starting at $(0,0)$ then crawls towards the one starting at $(10,0)$, which crawls towards the one starting at $(10,10)$, which crawls towards the one starting at $(0,10)$, which crawls towards the one starting at $(0,0)$.

1. If (x,y) is the position of the beetle that started at $(0,0)$ at some instant, what is the position of the beetle that started at $(10,0)$ at the same instant? *[2]*

Solution. The beetles move along congruent paths, allowing for their starting points and orientation. (To be precise, if you were to rotate the plane by a multiple of $90°$ about the centre of the initial square [*i.e.* the point $(5,5)$], in either direction, the curves followed by the beetles would lie directly atop one another.) It is not hard to see that it follows that as the beetle starting out at $(0,0)$ moves right x *cm* and up y *cm*, the beetle starting out at $(10,0)$ moves up x *cm* and left y *cm*. Thus, when the beetle starting at $(0,0)$ is at (x,y), the beetle starting at $(10,0)$ is at $(10-y,x)$. ■

2. Suppose (x,y) gives the position of the beetle that started at $(0,0)$. Use the fact that the beetle starting at $(0,0)$ crawls directly towards the one starting at $(10,0)$ at every instant to find an equation combining x, y, and $\frac{dy}{dx}$ that is satisfied by the curve followed by the beetle starting at $(0, 0)$. *[4]*

SOLUTION. Since the beetle starting at $(0, 0)$ crawls directly towards the one starting at $(10, 0)$ at every instant, at every instant the line joining its location to that of the beetle it is pursuing is tangent to the curve $y = y(x)$ it traces out.



The slope of this curve is therefore $\frac{dy}{dx}$. On the other hand, we can also compute the slope as rise over run, since, by our answer to bf 1 above, we know that when the beetle starting from $(10, 0)$ is at (x, y), the beetle starting at $(10, 0)$ is at $(10 - y, x)$. Thus

$$\frac{dy}{dx} = \text{slope of the line} = \frac{\text{rise}}{\text{run}} = \frac{x - y}{10 - y - x},$$

which is certainly "an equation combining x, y, and $\frac{dy}{dx}$ that is satisfied by the curve followed by the beetle starting at $(0, 0)$," as desired. ■

3. Use `Maple` to solve the equation you obtained in answering **2**. (Don't forget that the beetle starts at $(0, 0)$ for the most complete answer.) *[4]*

Hint: The `dsolve` command may come in handy here...

SOLUTION. Following the hint:

$$> \ dsolve\left(\left\{diff(y(x), x) = \frac{(x - y(x))}{(10 - y(x) - x)}, y(0) = 0, \mathrm{D}(y)(0) = 0\right\}\right)$$

$$y(x) = 5\tan\left(RootOf\left(4\_Z - 2\ln\left(\frac{1}{\cos(\_Z)^2}\right) - 4\ln(-5 + x) + 2\ln(2) + \pi + 4\ln(5) + 4\,\mathrm{I}\,\pi\right)\right) - \tan\left(RootOf\left(4\_Z - 2\ln\left(\frac{1}{\cos(\_Z)^2}\right) - 4\ln(-5 + x) + 2\ln(2) + \pi + 4\ln(5) + 4\,\mathrm{I}\,\pi\right)\right)x + 5 \quad \textbf{(1)}$$

It's pretty ugly – even if one has the tools to solve this by hand, having a piece of software do it is way easier ... ■

Bonus. Without using any of your work in **1–3**, determine how long it takes for all four beetles to meet after the start if they crawl at a constant rate of 1 cm/s. *[2]*

Hint: At any given instant each beetle is moving at right angles to the one it is pursuing and the one it is pursued by.

SOLUTION. I stole the problem from Martin Gardner's *Nine More Problems*†, which originally appeared in *Scientific American* in the late 1950's, in which it appears as problem 8 of 9. (I did phrase it slightly differently and changed inches to centimetres.) Here is the solution he gives:

> At any given instant the four bugs form the corners of a square which shrinks and rotates as the bugs move closer together. The path of each pursuer will therefore at all times be perpendicular to the path of the pursued. This tells us that as A, for example, approaches B, there is no component in B's motion which carries B toward or away from A. Consequently A will capture B in the same time that it would take if B had remained stationary. The length of each spiral path will be the same as the side of the square: 10 inches.

The key point is that the mutually perpendicular motion of the beetles implies that no beetle's motion has a component which carries it away or towards the beetle that is pursuing it. Thus each beetle closes on the beetle it is pursuing at its full 1 cm/s speed; since the are initially separated by by 10 cm, it follows that it takes 10 s before they all meet at the centre of the square. ■

† Specifically, from the version of this article in *Hexaflexagons and Other Mathematical Diversions*, by Martin Gardner, University of Chicago Press, Chicago, 1959, pp. 110–123.