TRENT UNIVERSITY

MATH 1101Y Test 2

11 February, 2011

Time: 50 minutes

Name: Steffi Graph — *I don't think we have a student with this name and number...*

STUDENT NUMBER: 01234567

Question ?	*Mark*
1	16
2	12
3	12
4	
Total	40

Eh? There are only three questions!

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Compute any *four* (4) of the integrals in parts **a-f**. *[16 = 4 × 4 each]*

a. $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$ **b.** $\displaystyle\int_0^{\pi/4} \sec(x)\tan(x)\,dx$ **c.** $\displaystyle\int_0^{\infty} e^{-x}\,dx$

d. $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$ **e.** $\displaystyle\int \frac{\cos(x)}{\sin(x)}\,dx$ **f.** $\displaystyle\int_1^{e} \ln(x)\,dx$

SOLUTIONS. **a.** We'll use the trig substitution $x = \tan(\theta)$, so $dx = \sec^2(\theta)\,d\theta$ and $\sqrt{x^2+1} = \sqrt{\tan^2(\theta)+1} = \sqrt{\sec^2(\theta)} = \sec(\theta)$.

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+1}}\,dx &= \int \frac{1}{\sec(\theta)}\sec^2(\theta)\,d\theta = \int \sec(\theta)\,d\theta = \ln\left(\tan(\theta)+\sec(\theta)\right)+C\\ &= \ln\left(x+\sqrt{x^2+1}\right)+C \qquad \square\end{aligned}$$

b. We'll use the substitution $u = \sec(x)$, so $du = \sec(x)\tan(x)\,dx$ and $\begin{array}{ccc} x & 0 & \pi/4 \\ u & 1 & \sqrt{2} \end{array}$. (Note that $\sec(\pi/4) = 1/\cos(\pi/4) = 1/\left(1/\sqrt{2}\right) = \sqrt{2}$.)

$$\int_0^{\pi/4} \sec(x)\tan(x)\,dx = \int_1^{\sqrt{2}} 1\,du = u\big|_1^{\sqrt{2}} = \sqrt{2}-1 \qquad \square$$

c. We'll use the substitution $w = -x$, so $dw = (-1)dx$ and $dx = (-1)dw$, and $\begin{array}{ccc} x & 0 & t \\ w & 0 & -t \end{array}$. Note that this is an improper integral, so we'll have to take a limit first.

$$\begin{aligned}\int_0^{\infty} e^{-x}\,dx &= \lim_{t\to\infty}\int_0^{t} e^{-x}\,dx = \lim_{t\to\infty}\int_0^{-t} e^{w}\,(-1)dw = \lim_{t\to\infty}\,(-1)e^{w}\big|_0^{-t}\\ &= \lim_{t\to\infty}\left[(-1)e^{-t}-(-1)e^{0}\right] = \lim_{t\to\infty}\left[-e^{-t}+1\right] = \lim_{t\to\infty}\left[1-\frac{1}{e^t}\right] = 1-0 = 1\end{aligned}$$

Note that $\frac{1}{e^t} \to 0$ as $t\to\infty$ since $e^t \to \infty$ as $t \to \infty$. $\square$

d. This is a job for partial fractions. Note first that $x^2+3x+2 = (x+1)(x+2)$. (This can be done by eyeballing, experimenting a bit, or using the quadratic formula to find the roots of x^2+3x+2. Calculators that can do some symbolic computation should be able to factor the quadratic too.) We must therefore have a partial fraction decomposition of the form

$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{A}{x+1}+\frac{B}{x+2}$$

for some constants A and B. It follows that

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1}+\frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)} = \frac{(A+B)x+(2A+B)}{(x+1)(x+2)},$$

so $A + B = 0$ and $2A + B = 1$. Then $A = (2A + B) - (A + B) = 1 - 0 = 1$ and $B = 0 - A = -1$.

We can now integrate at last; we'll use the substitutions $u = x + 1$ and $w = x + 2$, so $du = dx$ and $dw = dx$.

$$\begin{aligned}\int \frac{1}{x^2+3x+2}\,dx &= \int \left(\frac{1}{x+1} + \frac{-1}{x+2}\right) dx = \int \frac{1}{x+1}\,dx - \int \frac{1}{x+2}\,dx \\ &= \int \frac{1}{u}\,du - \int \frac{1}{w}\,dw = \ln(u) - \ln(w) + C \\ &= \ln(x+1) - \ln(x+2) + C \qquad \square\end{aligned}$$

e. We'll use the substitution $u = \sin(x)$, so $du = \cos(x)\,dx$.

$$\int \frac{\cos(x)}{\sin(x)}\,dx = \int \frac{1}{u}\,du = \ln(u) + C = \ln(\sin(x)) + C \qquad \square$$

f. We'll use integration by parts, with $u = \ln(x)$ and $v' = 1$, so $u' = \frac{1}{x}$ and $v = x$.

$$\begin{aligned}\int_1^e \ln(x)\,dx &= \int_1^e uv'\,dx = uv|_1^e - \int_1^e u'v\,dx = x\ln(x)|_1^e - \int_1^e \frac{1}{x}x\,dx \\ &= (e\ln(e) - 1\ln(1)) - \int_1^e 1\,dx = (e \cdot 1 - 1 \cdot 0) - x|_1^e = e - (e-1) = 1 \qquad \square\end{aligned}$$

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2. Do any *two* (2) of parts **a-e**. *[12 = 2 × 6 each]*

a. Compute $\displaystyle\int_1^2 \frac{x^3 - x^2 - x + 1}{x+1}\,dx$

b. Find the area between $y = \cos(x)$ and $y = \sin(x)$ for $0 \le x \le \frac{\pi}{2}$.

c. Which of $\displaystyle\int_\pi^{41} \arctan\left(\sqrt{x}\right)\,dx$ and $\displaystyle\int_\pi^{41} \arctan\left(x^2\right)\,dx$ is larger? Explain why.

d. Use the Right-hand Rule to compute $\displaystyle\int_1^2 x\,dx$.

e. Find the area of the region bounded by $y = 0$ and $y = \ln(x)$ for $0 < x \le 1$.

Solutions. **a.** This is a rational function whose numerator has degree greater than its denominator. Observe that

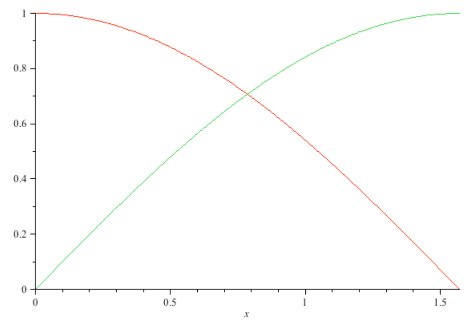
$$\begin{aligned}\frac{x^3 - x^2 - x + 1}{x+1} &= \frac{\left(x^3 - x\right) + \left(-x^2 + 1\right)}{x+1} = \frac{x\left(x^2-1\right) - 1\left(x^2-1\right)}{x+1} \\ &= \frac{(x-1)\left(x^2-1\right)}{x+1} = \frac{(x-1)(x-1)(x+1)}{x+1} = (x-1)^2,\end{aligned}$$

which we could also get by dividing $x+1$ into x^3-x^2-x+1 if we didn't spot the cheap bit of algebra above.

We can now integrate; we'll use the substitution $w = x-1$, so $dw = dx$, and we'll change limits accordingly: $\begin{array}{ccc} x & 1 & 2 \\ w & 0 & 1 \end{array}$. Thus:

$$\int_1^2 \frac{x^3-x^2-x+1}{x+1}\,dx = \int_1^2 (x-1)^2\,dx = \int_0^1 w^2\,dw = \left.\frac{w^3}{3}\right|_0^1 = \frac{1}{3}-\frac{0}{3} = \frac{1}{3} \qquad \Box$$

b. Recall what the graphs of $\cos(x)$ and $\sin(x)$ look like:

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plot( [cos(x), sin(x)], x=0..(1/2)*Pi );
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$\cos(0) = 1$ and $\sin(0) = 0$, but $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$; the graphs of the two functions cross each other at $x = \frac{\pi}{4}$, where both are equal to $1/\sqrt{2}$. The area between the curves is therefore:

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/4} (\cos(x)-\sin(x))\,dx + \int_{\pi/4}^{\pi/2} (\sin(x)-\cos(x))\,dx \\
&= (\sin(x)-(-\cos(x)))\big|_0^{\pi/4} + (-\cos(x)-\sin(x))\big|_{\pi/4}^{\pi/2} \\
&= \left(\sin\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{4}\right)\right) - (\sin(0)+\cos(0)) \\
&\quad + \left(-\cos\left(\frac{\pi}{2}\right)-\sin\left(\frac{\pi}{2}\right)\right) - \left(-\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)\right) \\
&= \left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right) - (0+1) + (-0-1) - \left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right) \\
&= \frac{4}{\sqrt{2}} - 2 = 2\sqrt{2}-2 = 2\left(\sqrt{2}-1\right) \qquad \Box
\end{aligned}$$

c. Note that the two definite integrals are the same except for the function of x being composed with arctan. As $\arctan(t)$ is an increasing function – its derivative, $\dfrac{1}{1+t^2}$, is positive for all t – and $\sqrt{x} < x^2$ for all $x > 1$, we must have $\arctan\left(\sqrt{x}\right) < \arctan\left(x^2\right)$ for all x in $[\pi, 41]$. It follows that $\displaystyle\int_\pi^{41} \arctan\left(\sqrt{x}\right)\,dx < \int_\pi^{41} \arctan\left(x^2\right)\,dx$. $\Box$

d. We throw the Right-hand Rule formula, $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{b-a}{n}\cdot f\left(a+i\frac{b-a}{n}\right)$, at the given definite integral and compute away. Note that $f(x) = x$ in this case.

$$\begin{aligned}
\int_1^2 x\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2-1}{n}\cdot\left(1+i\frac{2-1}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\cdot\left(1+\frac{i}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right) \\
&= \lim_{n\to\infty}\frac{1}{n}\left(\left[\sum_{i=1}^{n}1\right]+\left[\sum_{i=1}^{n}\frac{i}{n}\right]\right) = \lim_{n\to\infty}\frac{1}{n}\left(n+\left[\frac{1}{n}\sum_{i=1}^{n}i\right]\right) \\
&= \lim_{n\to\infty}\frac{1}{n}\left(n+\frac{1}{n}\cdot\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\frac{1}{n}\left(n+\frac{n+1}{2}\right) = \lim_{n\to\infty}\frac{1}{n}\left(\frac{3}{2}n+\frac{1}{2}\right) \\
&= \lim_{n\to\infty}\left(\frac{3}{2}+\frac{1}{2n}\right) = \frac{3}{2}+0 \quad \text{since } \frac{1}{2n}\to 0 \text{ as } n\to\infty. \ \square
\end{aligned}$$

e. Since $\ln(x) < 0$ for $0 < x < 1$, the area of the given region is just $\int_0^1 (0-\ln(x))\,dx = -\int_0^1 \ln(x)\,dx$. However, since $\ln(x)$ has an asymptote at $x = 0$, this is an improper integral, forcing us to do some additional work. To find the antiderivative of $\ln(x)$ itself, we will use integration by parts, with $u = \ln(x)$ and $v' = 1$, so $u' = \frac{1}{x}$ and $v = x$.

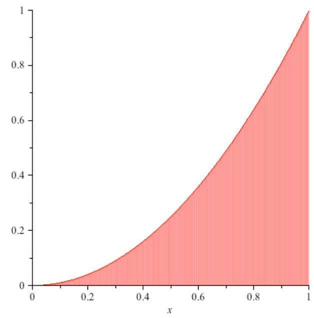
$$\begin{aligned}
\text{Area} &= -\int_0^1 \ln(x)\,dx = \lim_{t\to 0^+}\left(-\int_t^1 \ln(x)\,dx\right) = -\lim_{t\to 0^+}\int_t^1 \ln(x)\,dx \\
&= -\lim_{t\to 0^+}\left[x\ln(x)\big|_t^1 - \int_t^1 \frac{1}{x}x\,dx\right] = -\lim_{t\to 0^+}\left[1\ln(1) - t\ln(t) - \int_t^1 1\,dx\right] \\
&= -\lim_{t\to 0^+}\left[1\cdot 0 - t\ln(t) - x\big|_t^1\right] = -\lim_{t\to 0^+}\left[-t\ln(t)-(1-t)\right] \\
&= \lim_{t\to 0^+}\left[t\ln(t)+(1-t)\right] = \lim_{t\to 0^+}\frac{\ln(t)}{1/t} + \lim_{t\to 0^+}(1-t) \\
&\qquad \text{Use l'Hôpital's Rule since } \ln(t)\to-\infty \text{ and } \frac{1}{t}\to\infty \text{ as } t\to 0^+: \\
&= \left(\lim_{t\to 0^+}\frac{1/t}{-1/t^2}\right) + (1-0) = \left(\lim_{t\to 0^+} -t\right) + 1 = -0+1 = 1 \qquad \square
\end{aligned}$$

3. Do *one* (1) of parts **a** or **b**. *[12]*

a. Sketch the solid obtained by rotating the region bounded above by $y = x^2$ and below by $y = 0$, where $0 \leq x \leq 2$, about the y-axis, and find its volume.

b. Sketch the solid obtained by rotating the region bounded above by $y = x^2$ and below by $y = 0$, where $0 \leq x \leq 2$, about the x-axis, and find its volume.

Solutions. Note that the region being rotated is the same in both **a** and **b**; they differ in the axis about which the region is rotated.

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plot(x^2,x=0..1,color="Red",filled=[color="Red",transparency=.5])
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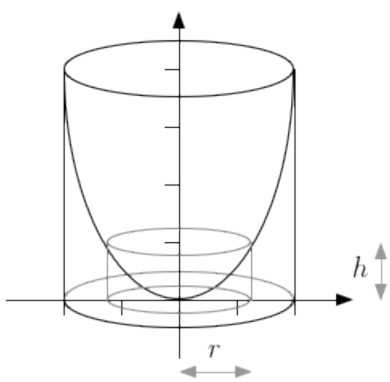
Solution to **a**. Here is a crude sketch of the solid with a generic cylindrical shell.



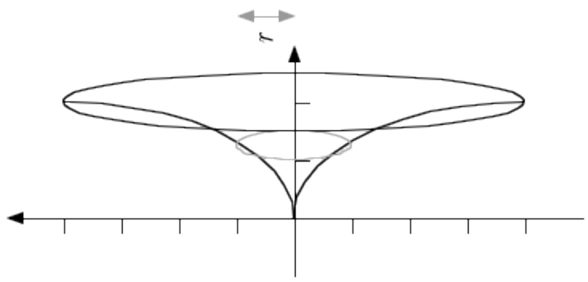
The solid with a cylindrical shell.

We will find the volume of the solid using cylindrical shells. Note that since we rotated the region about the y-axis, we will have to integrate with respect to x if we're using shells. Looking at the diagram, it is easy to see that the radius of the cylindrical shell that comes from rotating the vertical cross-section at x of the original region is just going to be $r = x - 0 = x$. It is also easy to see that its height, which is the length of the vertical cross-section at x of the original region, is going to be $h = x^2 - 0 = x^2$. The limits of integration will come from the possible x values in the original region, *i.e.* $0 \leq x \leq 2$.

Thus the volume of the solid is:

$$\begin{aligned}\text{Volume} &= \int_0^2 2\pi r h\, dx = \int_0^2 2\pi x x^2\, dx = 2\pi \int_0^2 x^3\, dx \\ &= 2\pi \frac{x^4}{4}\Big|_0^2 = 2\pi\left(\frac{2^4}{4} - \frac{0^4}{4}\right) = 2\pi\left(\frac{16}{4} - 0\right) = 8\pi \qquad \square\end{aligned}$$

SOLUTION TO **b**. Here is a crude sketch of the solid with a generic disk.



The solid with a disk.
Rotate picture 90° *clockwise!*

We will find the volume of the solid using disks. Note that since we rotated the region about the x-axis, we will have to integrate with respect to x if we're using disks. Looking at the diagram, it is easy to see that the radius of the disk that comes from rotating the vertical cross-section at x of the original region is just going to be the length of that vertical cross-section, namely $r = x^2 - 0 = x^2$. Note that the disk has no hole because the x-axis forms part of the boundary of the give region, so we needn't worry about the inner radius: it is always 0. The limits of integration will come from the possible x values in the original region, *i.e.* $0 \leq x \leq 2$.

Thus the volume of the solid is:

$$\begin{aligned}\text{Volume} &= \int_0^2 \pi r^2\, dx = \pi \int_0^2 \left(x^2\right)^2\, dx = \pi \int_0^2 x^4\, dx \\ &= \pi\, \frac{x^5}{5}\Big|_0^2 = \pi\left(\frac{2^5}{5} - \frac{0^5}{5}\right) = \frac{32}{5}\pi \qquad \square\end{aligned}$$

[Total = 40]