TRENT UNIVERSITY

MATH 1101Y Test 1

19 November, 2010

Time: 50 minutes

Name: Steffi Graph

STUDENT NUMBER: 01234567

Question	Mark
1	____
2	____
3	____
4	____
Total	____

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Find $\frac{dy}{dx}$ in any *three* (3) of **a**-**e**. *[12 = 3 × 4 each]*

a. $y = x^x$ **b.** $y = \frac{1}{1+x^2}$ **c.** $y = \cos\left(\sqrt{x}\right)$ **d.** $y^2 + x = 1$ **e.** $y = x^2 e^{-x}$

SOLUTIONS. **a.** $y = x^x = \left(e^{\ln(x)}\right)^x = e^{x\ln(x)}$ so, using the Chain and Product Rules:

$$\frac{dy}{dx} = \frac{d}{dx}x^x = \frac{d}{dx}e^{x\ln(x)} = e^{x\ln(x)} \cdot \frac{d}{dx}\left(x\ln(x)\right) = e^{x\ln(x)} \cdot \left[\left(\frac{d}{dx}x\right) \cdot \ln(x) + x \cdot \frac{d}{dx}\ln(x)\right]$$
$$= e^{x\ln(x)} \cdot \left[1 \cdot \ln(x) + x \cdot \frac{1}{x}\right] = x^x \cdot \left(\ln(x) + 1\right)$$

This can also be done using logarithmic differentiation. □

b. Using the Quotient Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{\left(\frac{d}{dx}1\right) \cdot \left(1+x^2\right) - 1 \cdot \frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$
$$= \frac{0 \cdot \left(1+x^2\right) - 1 \cdot (2x)}{\left(1+x^2\right)^2} = \frac{-2x}{\left(1+x^2\right)^2} \qquad \square$$

c. Using the Chain Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\cos\left(\sqrt{x}\right) = -\sin\left(\sqrt{x}\right) \cdot \frac{d}{dx}\sqrt{x} = -\sin\left(\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}} = \frac{-\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Recall that $\sqrt{x} = x^{1/2}$, so, using the Power Rule, $\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} - \frac{1}{2\sqrt{x}}$. □

d. Using implicit differentiation and the Chain Rule:

$$y^2 + x = 1 \quad \Rightarrow \quad \frac{d}{dx}\left(y^2 + x\right) = \frac{d}{dx}1 \quad \Rightarrow \quad 2y\frac{dy}{dx} + 1 = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{2y}$$

One could also solve for y in terms of x in the original equation and then differentiate. □

e. Using the Product and Chain Rules:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2e^{-x}\right) = \left(\frac{d}{dx}x^2\right) \cdot e^{-x} + x^2 \cdot \left(\frac{d}{dx}e^{-x}\right) = 2xe^{-x} + x^2e^{-x} \cdot \left(\frac{d}{dx}(-x)\right)$$
$$= 2xe^{-x} + x^2e^{-x} \cdot (-1) = \left(2x - x^2\right)e^{-x} = x\left(2-x\right)e^{-x} \qquad \square$$

2. Do any *two* (2) of **a-d**. *[10 = 2 × 5 each]*

a. Use the limit definition of the derivative to compute $f'(0)$ for $f(x) = x^2 - 3x + \pi$.

b. Suppose $f(x) = \dfrac{x}{\sin(x)}$ for $x \neq 0$. What would $f(0)$ have to be to make $f(x)$ continuous at $a = 0$?

c. Find the equation of the tangent line to $y = x^2$ at the point $(2, 4)$.

d. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 1} (2x + 3) = 5$.

SOLUTIONS. **a.** Plug into the definition and chug away:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\left(h^2 - 3h + \pi\right) - \left(0^2 - 3 \cdot 0 + \pi\right)}{h} \\ &= \lim_{h \to 0} \frac{h^2 - 3h + \pi - \pi}{h} = \lim_{h \to 0} \frac{h^2 - 3h + 0}{h} = \lim_{h \to 0} (h - 3) = 0 - 3 = -3 \qquad \square \end{aligned}$$

b. To make $f(x)$ continuous at $a = 0$, we need to make $f(0) = \lim_{x \to 0} f(x)$, so we have to compute the limit:

$$\begin{aligned} \lim_{x \to 0} f(x) &= \lim_{x \to 0} \frac{x}{\sin(x)} && \text{This we compute using l'Hôpital's Rule, since both } x \to 0 \text{ and } \sin(x) \to 0 \text{ as } x \to 0. \\ &= \lim_{x \to 0} \frac{\frac{d}{dx} x}{\frac{d}{dx} \sin(x)} = \lim_{x \to 0} \frac{1}{\cos(x)} = \frac{1}{1} = 1 && \text{Since } \cos(x) \to 1 \text{ as } x \to 0. \end{aligned}$$

Thus we need to make $f(0) = 1$ to have $f(x)$ be continuous at $a = 1$. $\square$

c. The slope of the tangent line to $y = x^2$ at $(2, 4)$ is given by $\dfrac{dy}{dx} = \dfrac{d}{dx} x^2 = 2x$ evaluated at $x = 2$: $m = \left.\dfrac{dy}{dx}\right|_{x=2} = 2x|_{x=2} = 2 \cdot 2 = 4$. The equation of the line is therefore $y = 4x + b$ for some b. To find b, plug the coordinates of the point $(2, 4)$ in for x and y in the equation of the line and solve for b: $4 = 4 \cdot 2 + b$, so $b = 4 - 8 = -4$.

Thus the equation of the tangent line to $y = x^2$ at $(2, 4)$ is $y = 4x - 4$. $\square$

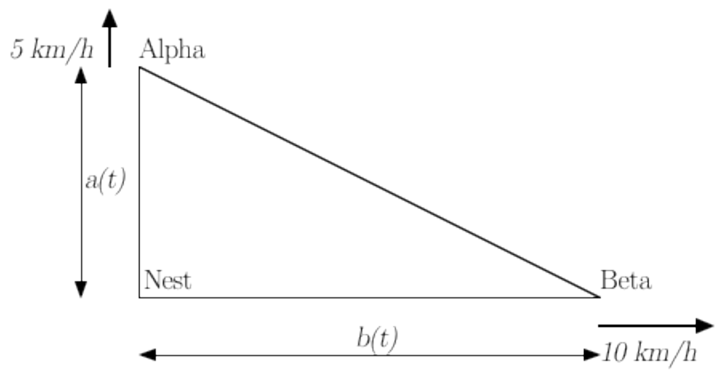
d. To verify that $\lim_{x \to 1} (2x + 3) = 5$, we need to show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $|(2x + 3) - 5| < \varepsilon$. As usual, we reverse-engineer the required δ from what we need to achieve:

$$|(2x + 3) - 5| < \varepsilon \quad \Leftrightarrow \quad |2x - 2| < \varepsilon \quad \Leftrightarrow \quad 2|x - 1| < \varepsilon \quad \Leftrightarrow \quad |x - 1| < \frac{\varepsilon}{2}$$

It follows that $\delta = \dfrac{\varepsilon}{2}$ does the job: if $|x - 1| < \delta = \frac{\varepsilon}{2}$, we can traverse the chain of equivalences above backwards to obtain $|(2x + 3) - 5| < \varepsilon$, as required.

Thus $\lim_{x \to 1} (2x + 3) = 5$. $\square$

3. Birds Alpha and Beta leave their nest at the same time, with Alpha flying due north at 5 *km/h* and Beta flying due east at 10 *km/h*. How is the area of the triangle formed by their respective positions and the nest changing 1 *h* after their departure? *[8]*



SOLUTION. Note that after 1 h, Alpha and Beta will have flown 5 km and 10 km, respectively.

Let $a(t)$ and $b(t)$ be the distances that birds Alpha and Beta, respectively, are from the nest at time t. Then most of the given information can be summarized as follows: $a(0) = b(0) = 0$, $a(1) = 5$, $b(1) = 10$, $a'(t) = 5$, and $b'(t) = 10$. Since the birds fly north and east, respectively, their positions and the position of the nest form a right triangle with base $b(t)$ and height $a(t)$ at each instant; the area of this triangle is therefore $A(t) = \frac{1}{2}a(t)b(t)$. We want to know what $A'(t)$ is at $t = 1$.

Using the Product Rule:

$$A'(t) = \frac{d}{dt}\left[\frac{1}{2}a(t)b(t)\right] = \frac{1}{2}\left[a'(t)b(t) + a(t)b'(t)\right]$$

Hence

$$\begin{aligned} A'(1) &= \frac{1}{2}\left[a'(1)b(1) + a(1)b'(1)\right] \\ &= \frac{1}{2}\left[5\ km/h \cdot 10\ km + 5\ km \cdot 10\ km/h\right] \\ &= \frac{1}{2}100\ km^2/h = 50\ km^2/h\,, \end{aligned}$$

i.e. the area of the triangle formed by the birds' respective positions and the nest is increasing at a rate of 50 km^2/h one hour after their departure from the nest. □

4. Find the domain and all intercepts, maxima and minima, and vertical and horizontal asymptotes of $f(x) = \dfrac{x^2+2}{x^2+1}$ and sketch its graph based on this information. *[10]*

SOLUTION. We run through the checklist:

Domain. $f(x) = \frac{x^2+2}{x^2+1}$ makes sense for all possible x – note that since $x^2 \geq 0$, the denominator is always $\geq 1 > 0$ – so the domain of $f(x)$ is $(-\infty, \infty)$.

Intercepts. $f(0) = \frac{0^2+2}{0^2+1} = \frac{2}{1} = 2$, so the y-intercept is $(0,2)$. Since $x^2+2 \geq 2$ for all x – since, again, $x^2 \geq 0$ – $f(x)$ is never 0, so $f(x)$ has no x-intercepts.

Maxima and minima. There are no endpoints to worry about, so all we need to do is check what happens around critical points. Using the Quotient Rule,

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}\left(x^2+2\right)\cdot\left(x^2+1\right) - \left(x^2+2\right)\cdot\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2} \\ &= \frac{2x\cdot\left(x^2+1\right) - \left(x^2+2\right)\cdot 2x}{\left(x^2+1\right)^2} = \frac{-2x}{\left(x^2+1\right)^2}, \end{aligned}$$

which $= 0$ when $x = 0$, > 0 when $x < 0$, and < 0 when $x > 0$. Note that there are no points where $f'(x)$ is undefined, since $\left(x^2+1\right)^2 \geq 1 > 0$ for all x. We build the usual table:

x	$(-\infty,0)$	0	$(0,\infty)$
$f'(x)$	$+$	0	$-$
$f(x)$	$\uparrow$	max	$\downarrow$

Since $f(x)$ is increasing to the left of 0 and decreasing to the right of 0, the critical point 0 (also the y-intercept!) is a maximum. Note that there are no minimum points.

Vertical asymptotes. Since $f(x)$ is defined for all x and continuous (being a rational function) wherever it is defined, $f(x)$ has no vertical asymptotes.

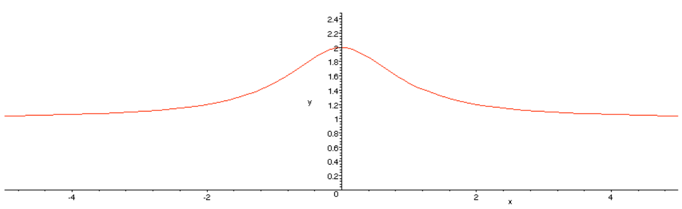
Horizontal asymptotes. We need to check what $f(x)$ does as $x \to \pm\infty$:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2+2}{x^2+1} = \lim_{x\to+\infty} \frac{x^2+2}{x^2+1}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to+\infty} \frac{1+2/x^2}{1+1/x^2} = \frac{1+0}{1+0} = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+2}{x^2+1} = \lim_{x\to-\infty} \frac{x^2+2}{x^2+1}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{1+2/x^2}{1+1/x^2} = \frac{1+0}{1+0} = 1$$

It follows that $f(x)$ has $y = 1$ as its horizontal asymptote in both directions.

The graph. `plot((x^2+2)/(x^2+1),x=-5..5,y=0..2.5);` in `Maple` gives:



That's that! $\square$

[Total = 40]