Math 1100 — Calculus, Quiz #9B — 2009-12-10

Let $f(x) := \dfrac{x^2}{x-2}$, for all $x \in \mathbb{R}$ where this formula makes sense.

(5) 1. What is the domain of f?

Solution: The domain is $(-\infty, 2) \sqcup (2, \infty)$, because the formula for f makes sense for all $x \in \mathbb{R}$ except $x = 2$ (where the denominator is zero). $\square$

(10) 2. What are the x-intercepts and y-intercepts of f?

Solution: For all $x \in \mathbb{R}$, we have

$$\left(\tfrac{x^2}{x-2} = 0\right) \iff \left(x^2 = 0\right) \iff \left(x = 0\right).$$

Thus, $\boxed{\text{the sole } x\text{-intercept is at } (0,0), \text{ which is also the } y\text{-intercept.}}$ $\square$

(10) 3. What symmetries does f have? Is it odd? even? Periodic?

Solution: f is neither even nor odd. To see this, note that

$$f(-x) \quad = \quad \frac{(-x)^2}{(-x)-2} \quad = \quad \frac{x^2}{-x-2} \quad = \quad -\frac{x^2}{x+2} \quad \neq \quad \pm f(x).$$

Furthermore, f is not periodic. $\square$

(20) 4. Find all vertical, horizontal, and slant asymptotes of f.

Solution: Since the formula for f breaks down at $x = 2$, there is potentially a vertical asymptote there. We have:

$$\lim_{x \searrow 2} \frac{x^2}{x-2} \quad = \quad \infty \quad \text{and} \quad \lim_{x \nearrow 2} \frac{x^2}{x-2} \quad = \quad -\infty$$

To see this, note that the numerator x^2 is positive for all x close to 2, while the denominator $x - 2$ converges to 0 as $x \to 2$. Furthermore, $x - 2 > 0$ if $x > 2$, while $x - 2 < 0$ if $x < 2$.

Next, observe that $f = \frac{\text{quadratic}}{\text{linear}}$, so f should grow roughly linearly as $x \to \pm\infty$. In other words, we expect a slant asymptote. To identify the slant asymptote, we perform polynomial long division, to obtain:

$$x^2 \;=\; (x-2)(x+2) + 4$$

Thus, we suspect a slant asymptote along the line $y = x + 2$. Let's confirm this:

$$\frac{x^2}{x-2} - (x+2) \quad = \quad \frac{x^2}{x-2} - \frac{(x+2)(x-2)}{x-2} \quad = \quad \frac{x^2 - (x^2-4)}{x-2} \quad = \quad \frac{4}{x-2}$$

Thus, $\displaystyle \lim_{x \to \pm\infty} \frac{x^2}{x-2} - (x+2) \quad = \quad \lim_{x \to \pm\infty} \frac{4}{x-2} \quad = \quad 0.$

Thus, $\boxed{f \text{ has a slant asymptote along the line } y = x + 2, \text{ as } x \to \pm\infty.}$ $\square$

(15) 5. Compute f'. Use this to find all intervals where f is increasing/decreasing.

Solution: The quotient rule says

$$f'(x) = \frac{(x-2)\cdot 2x - x^2\cdot(1)}{(x-2)^2} = \frac{2x^2-4x-x^2}{(x-2)^2} = \frac{x^2-4x}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}.$$

Now, f' can only change sign at places where it is zero or has a discontinuity. The roots of f' are the roots of the numerator $x(x-4)$, which are at 0 and 4. Also, f' has a discontinuity at 2 (where the denominator is 0), We make a table:

Interval	x	$(x-4)$	$(x-2)^2$	$f'(x)$	f is....
$(-\infty,0)$	$(-)$	$(-)$	$(+)$	$(+)$	increasing
$(0,2)$	$(+)$	$(-)$	$(+)$	$(-)$	decreasing
$(2,4)$	$(+)$	$(-)$	$(+)$	$(-)$	decreasing
$(4,\infty)$	$(+)$	$(+)$	$(+)$	$(+)$	increasing

□

(10) 6. Find all local maxima and minima of f.

Solution: Fermat's theorem says that extrema can only occur at the critical points of f. Now, f is differentiable everywhere in its domain $(-\infty,2)\sqcup(2,\infty)$, so

$$\Big(x \text{ is a critical point of } f\Big) \iff \Big(f'(x)=0\Big) \iff \Big(x\cdot(x-4)=0\Big) \iff \Big(x=0 \text{ or } x=4\Big)$$

From the previous question, we see that f is increasing to the left of 0 and decreasing to the right of 0 —thus, 0 is a *local maximum*. On the other hand, f is decreasing to the left of 4 and increasing to the right of 4 —thus, 4 is a *local minimum*. Finally, we observe that $f(0)=0$ and $f(4)=8$. □

(15) 7. Compute f''. Use this to identify the intervals of concavity and inflection points.

Solution: We have seen that

$$f'(x) = \frac{x^2-4x}{(x-2)^2}.$$

Thus,
$$f''(x) = \frac{(x-2)^2\cdot(2x-4)-(x^2-4x)\cdot 2(x-2)}{(x-2)^4} = \frac{(x-2)\cdot(2x-4)-2(x^2-4x)}{(x-2)^3}$$
$$= \frac{(2x^2-8x+8)-(2x^2-8x)}{(x-2)^3} = \frac{8}{(x-2)^3}.$$

Thus,

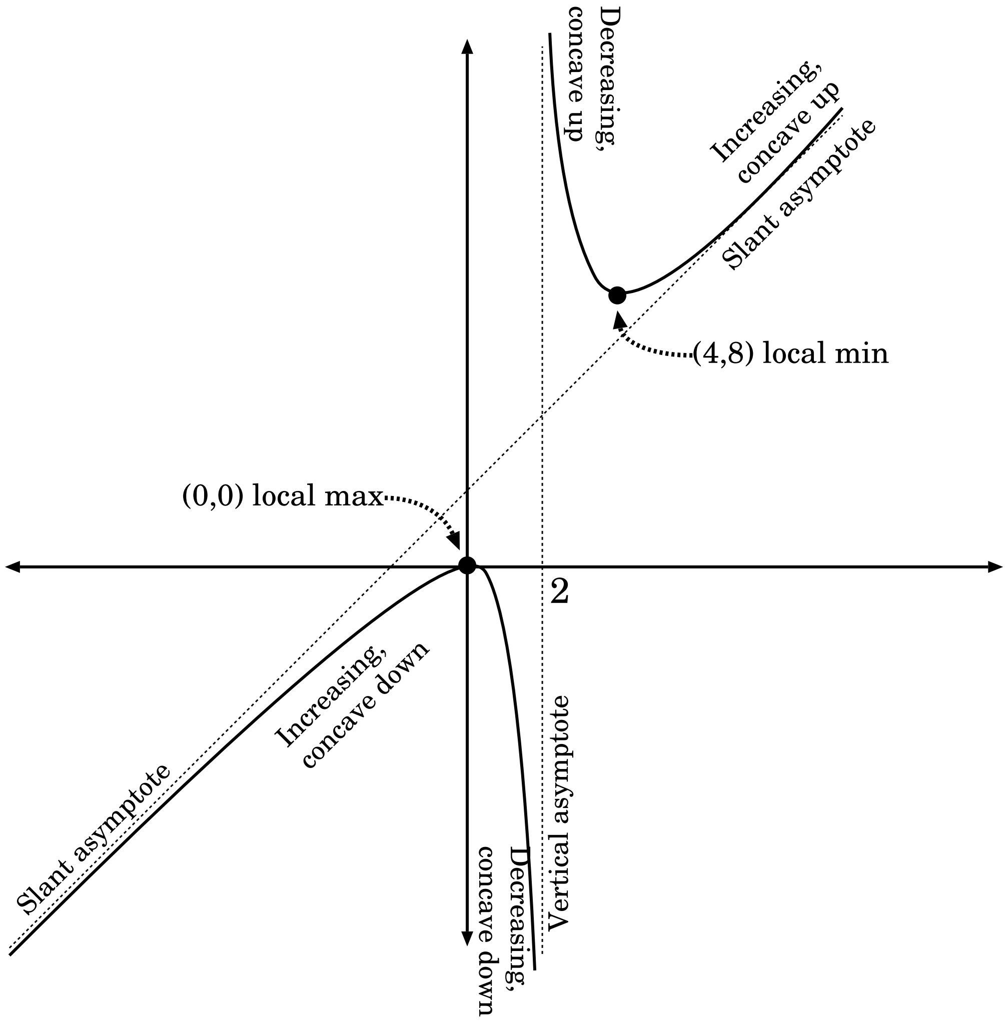
$$\Big(f \text{ is concave-up}\Big) \iff \Big(f''(x)>0\Big) \iff \Big((x-2)^3>0\Big) \iff \Big(x-2>0\Big) \iff \Big(x>2\Big).$$

Likewise, $\Big(f \text{ is concave-down}\Big) \iff \Big(x<2\Big)$.

Finally, note that 2 is *not* an inflection point (even though the concavity changes there) —it is a vertical asymptote.

□

(15) 8. Use all of the above information to sketch the curve of f on its domain.



Solution:

□