Math 1100 — Calculus, Quiz #16A — 2010-03-15

The following formulae might be useful:

$$\begin{aligned}
\int \tan(x)\ \mathrm{d}x &= \ln|\sec(x)| + C. & \int \sec(x)\ \mathrm{d}x &= \ln|\sec(x)+\tan(x)| + C.\\
\int \cot(x)\ \mathrm{d}x &= \ln|\sin(x)| + C. & \int \csc(x)\ \mathrm{d}x &= \ln|\csc(x)-\cot(x)| + C.\\
\int \sec(x)^2\ \mathrm{d}x &= \tan(x) + C. & \int \sec(x)\tan(x)\ \mathrm{d}x &= \sec(x) + C.\\
\int \csc(x)^2\ \mathrm{d}x &= -\cot(x) + C. & \int \csc(x)\cot(x)\ \mathrm{d}x &= -\csc(x) + C.
\end{aligned}$$

(50) 1. Let $f(x) = \ln[\sin(x)]$. Compute the *arc-length* of the graph of f between the points $x = \pi/4$ and $x = \pi/2$.

Solution: If $f(x) = \ln[\sin(x)]$, then

$$\begin{aligned}
f'(x) &= \frac{\sin'(x)}{\sin(x)} = \frac{\cos(x)}{\sin(x)} = \cot(x).\\
\text{Thus,}\quad \sqrt{1+f'(x)^2} &= \sqrt{1+\cot(x)^2} = \sqrt{\csc(x)^2} = |\csc(x)| \underset{(*)}{=} \csc(x).\\
\text{Thus,}\quad \text{arc-length} &= \int_{\pi/4}^{\pi/2} \sqrt{1+f'(x)^2}\ \mathrm{d}x = \int_{\pi/4}^{\pi/2} \csc(x)\ \mathrm{d}x\\
&= \ln|\csc(x)-\cot(x)|\ \Big|_{x=\pi/4}^{x=\pi/2}\\
&= \ln|\csc(\pi/2)-\cot(\pi/2)| - \ln|\csc(\pi/4)-\cot(\pi/4)|\\
&= \ln|1-0| - \ln\left|\sqrt{2}-1\right| = \boxed{-\ln\left(\sqrt{2}-1\right).}
\end{aligned}$$

Here $(*)$ uses the fact that $\csc(\theta) = 1/\sin(\theta) > 0$ for $\theta \in [\pi/4, \pi/2]$. □

(50) 2. Let $\mathcal{C}$ be the graph of the function $f(x) = \sqrt{1+4x}$ for $x \in [0,1]$. Let $\mathcal{S}$ be the surface of revolution obtained by rotating $\mathcal{C}$ around the x axis. Compute the *surface area* of $\mathcal{S}$.

Solution: If $f(x) = \sqrt{1+4x}$, then

$$\begin{aligned}
f'(x) &= \frac{4}{2\sqrt{1+4x}} = \frac{2}{\sqrt{1+4x}}.\\
\text{Thus,}\quad f'(x)^2 &= \frac{4}{1+4x}.\\
\text{Thus,}\quad f(x)\cdot\sqrt{1+f'(x)^2} &= \sqrt{1+4x}\cdot\sqrt{1+\frac{4}{1+4x}} = \sqrt{(1+4x)+\frac{4(1+4x)}{1+4x}}\\
&= \sqrt{1+4x+4} = \sqrt{5+4x}.\\
\text{Thus,}\quad \text{Area} &= 2\pi\int_0^1 f(x)\cdot\sqrt{1+f'(x)^2}\ \mathrm{d}x = 2\pi\int_0^1 \sqrt{5+4x}\ \mathrm{d}x\\
&\underset{(*)}{=} \frac{2\pi}{4}\int_5^9 \sqrt{u}\ \mathrm{d}u = \frac{\pi}{2}\frac{2\,u^{3/2}}{3}\Big|_{u=5}^{u=9} = \frac{\pi}{3}\left(9^{3/2}-5^{3/2}\right)\\
&= \boxed{\frac{\pi}{3}\left(27-5^{3/2}\right).}
\end{aligned}$$

Here $(*)$ is the change of variables $u = 5 + 4x$, so that $\mathrm{d}u = 4\ \mathrm{d}x$. $\square$