MATH 1101Y 2009 Quiz 17 (a)

1. (1 pts) Identify the curve by finding a Cartesian equation for the curve

$$r = 2\cos\theta - 4\sin\theta$$

Solution: We have

$$\begin{aligned} r^2 &= 2r\cos\theta - 4r\sin\theta \\ x^2 + y^2 &= 2x - 4y \end{aligned}$$

$$\begin{aligned} x^2 - 2x + y^2 + 4y &= 0 \\ x^2 - 2x + 1 + y^2 + 4y + 4 &= 5. \end{aligned}$$

The Cartesian equation for the curve is

$$(x-1)^2 + (y+2)^2 = \left(\sqrt{5}\right)^2.$$

The curve represents a circle with centre at $(1, -2)$ and radius $\sqrt{5}$. □

2. (2 pts) Find the (x, y)-coordinates of the points on the curve $r = 3\cos\theta$ where the tangent line is horizontal or vertical.

Solution: Since

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}},$$

and $y = r\sin\theta$, $x = r\cos\theta$, we have

$$\begin{aligned} \frac{dy}{d\theta} &= (3\cos\theta\sin\theta)' \\ &= -3\sin^2\theta + 3\cos^2\theta \end{aligned}$$

$$\begin{aligned} \frac{dy}{d\theta} &= 0 \Leftrightarrow -3\sin^2\theta + 3\cos^2\theta = 0 \\ &\Leftrightarrow \sin^2\theta = \cos^2\theta \\ &\Leftrightarrow (\tan\theta)^2 = 1 \Leftrightarrow \tan\theta = \pm 1. \end{aligned}$$

Therefore, $\frac{dy}{d\theta} = 0$ when $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$. Similarly, we have

$$\begin{aligned} \frac{dx}{d\theta} &= \left(3\cos^2\theta\right)' \\ &= -6\cos\theta\sin\theta \end{aligned}$$

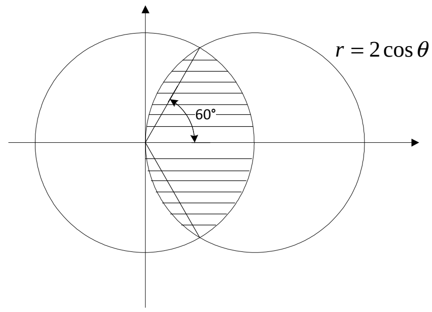
$$\begin{aligned} \frac{dx}{d\theta} &= 0 \Leftrightarrow -6\cos\theta\sin\theta \\ &\Leftrightarrow \cos\theta = 0 \text{ or } \sin\theta = 0. \end{aligned}$$

Therefore, $\frac{dx}{d\theta} = 0$ when $\theta = 0, \pi, \pm\frac{\pi}{2}$.

The curve has a horizontal tangent line at $\left(\frac{3}{2}, \frac{3}{2}\right)$ and $\left(\frac{3}{2}, -\frac{3}{2}\right)$; a vertical tangent line at $(3, 0)$ and $(0, 0)$. □

3. (2 pts) Set up an integral that represents the area of the region that is inside both $r = 2\cos\theta$ and $r = 1$. Do not evaluate this integral.

Solution:



First we find where these two curves meet.

$$\begin{aligned} 2\cos\theta &= 1 \\ \cos\theta &= \frac{1}{2} \end{aligned}$$

$$\theta = \pm\frac{\pi}{3}$$

Since the region is symmetric about the x-axis, we will calculate the area of one quater of it then multiply it by 2.

$$A = 2\left(\int_0^{\frac{\pi}{3}} \frac{1}{2}(1)^2\,d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta)^2\,d\theta\right)$$

$\square$