Mathematics 110 – Calculus of one variable
TRENT UNIVERSITY, 2002-2003

Sollutions for Test #2 - Section A

1. Compute any *three* of the integrals in parts **a**-**f**. *[36 = 3 × 12 each]*

a. $\displaystyle\int_0^1 \arctan(x)\,dx$ **b.** $\displaystyle\int_e^{e^e} \frac{\ln(\ln(x))}{x\ln(x)}\,dx$ **c.** $\displaystyle\int \frac{1}{x^2-3x+2}\,dx$

d. $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$ **e.** $\displaystyle\int_0^{\pi/4} \sec^4(x)\,dx$ **f.** $\displaystyle\int_2^{\infty} \frac{1}{x^4}\,dx$

Solutions:
a. We'll use integration by parts, with $u = \arctan(x)$ and $dv = dx$, so $du = \dfrac{1}{1+x^2}\,dx$ and $v = x$.

$$\int_0^1 \arctan(x)\,dx = x\arctan(x)\Big|_0^1 = 1\cdot\arctan(1) - 0\cdot\arctan(0) = 1\cdot\frac{\pi}{4} - 0\cdot 0 = \frac{\pi}{4} \quad \blacksquare$$

b. Here we'll substitute whole hog, namely $u = \ln(\ln(x))$. Note that then

$$\frac{du}{dx} = \frac{1}{\ln(x)}\cdot\frac{d}{dx}\ln(x) = \frac{1}{\ln(x)}\cdot\frac{1}{x}\ .$$

Also, when $x = e$,

$$u = \ln(\ln(e)) = \ln(1) = 0\,,$$

and when $x = e^e$,

$$u = \ln(\ln(e^e)) = \ln(e\ln(e)) = \ln(e\cdot 1) = \ln(e) = 1\ .$$

Hence

$$\int_e^{e^e} \frac{\ln(\ln(x))}{x\ln(x)}\,dx = \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}1^2 - \frac{1}{2}0^2 = \frac{1}{2}\,. \quad \blacksquare$$

c. This one is a job for integration by partial fractions. Since $x^2-3x+2 = (x-2)(x-1)$,

$$\frac{1}{x^2-3x+2} = \frac{1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}\,,$$

where $1 = A(x-1)+B(x-2) = (A+B)x + (-A-2B)$. It follows that $0 = A+B$, so $A = -B$, and $1 = -A-2B = B-2B = -B$, so $B = -1$ and $A = 1$. Then:

$$\int \frac{1}{x^2-3x+2}\,dx = \int\left(\frac{1}{x-2} - \frac{1}{x-1}\right)dx = \int\frac{1}{x-2}\,dx - \int\frac{1}{x-1}\,dx$$
$$= \ln(x-2) - \ln(x-1) + C = \ln\left(\frac{x-2}{x-1}\right) + C \quad \blacksquare$$

d. Note that $x^2 + 2x + 2 = x^2 + 2x + 1 + 1 = (x + 1)^2 + 1$. We will use the substitution $u = x + 1$, so $du = dx$, and the fact that $\frac{d}{du} \arctan(u) = \frac{1}{u^2 + 1}$:

$$\int \frac{1}{x^2 + 2x + 1}\, dx = \int \frac{1}{(x + 1)^2 + 1}\, dx = \int \frac{1}{u^2 + 1}\, du = \arctan(u) + C \quad \blacksquare$$

e. Here we will use the fact that $\sec^2(x) = 1 + \tan^2(x)$ and the substitution $u = \tan(x)$, so $du = \sec^2(x)\, dx$. Note that when $x = 0$, $u = \tan(0) = 0$, and when $x = \frac{\pi}{4}$, $u = \tan\left(\frac{\pi}{4}\right) = 1$.

$$\begin{aligned}
\int_0^{\pi/4} \sec^4(x)\, dx &= \int_0^{\pi/4} \sec^2(x) \sec^2(x)\, dx \\
&= \int_0^{\pi/4} \left(1 + \tan^2(x)\right) \sec^2(x)\, dx = \int_0^1 \left(1 + u^2\right)\, du \\
&= \left(u + \frac{1}{3}u^3\right)\Big|_0^1 = \left(1 + \frac{1}{3} \cdot 1^3\right) - \left(0 + \frac{1}{3} \cdot 0^3\right) = \frac{4}{3} \quad \blacksquare
\end{aligned}$$

f. Since it has an infinity in one of the limits of integration, this is obviously an improper integral.

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x^4}\, dx = \int_2^{\infty} x^{-4}\, dx &= \lim_{t\to\infty} \int_2^t x^{-4}\, dx \\
&= \lim_{t\to\infty} \frac{x^{-3}}{-3}\Big|_2^t = \lim_{t\to\infty} \left(\frac{-1}{3t^3} - \frac{-1}{3 \cdot 2^3}\right) \\
&= \lim_{t\to\infty} \left(\frac{1}{24} - \frac{1}{3t^3}\right) = \frac{1}{24} - 0 = \frac{1}{24}
\end{aligned}$$

Note that as $t \to \infty$, $t^3 \to \infty$ as well, so $\frac{1}{3t^3} \to 0$. $\blacksquare$

2. Do any *two* of parts **a**-**d**. *[24 = 2 × 12 each]*

a. Compute $\frac{d}{dx}\left(\int_1^{\cos(x)} \arccos(t)\, dt\right)$.

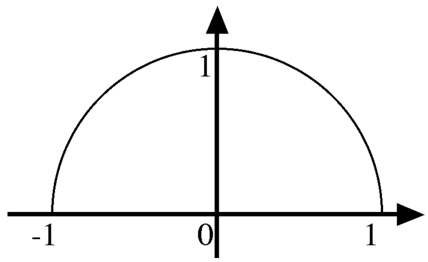
Solution. Let $y = \int_1^{\cos(x)} \arccos(t)\, dt$ and $u = \cos(x)$. Then, using the Chain Rule, the Fundamental Theorem of Calculus, and the fact that $\arccos(\cos(x)) = x$:

$$\begin{aligned}
\frac{d}{dx}\left(\int_1^{\cos(x)} \arccos(t)\, dt\right) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} &= \frac{d}{du}\left(\int_1^u \arccos(t)\, dt\right) \cdot \frac{du}{dx} \\
&= \arccos(u) \cdot \frac{du}{dx} = \arccos(\cos(x)) \cdot \frac{d}{dx} \cos(x) \\
&= x\left(-\sin(x)\right) = -x \sin(x) \quad \blacksquare
\end{aligned}$$

b. Give both a description and a sketch of the region whose area is computed by the integral $\int_{-1}^{1} \sqrt{1-x^2}\, dx$.

Solution. Note that if $y = \sqrt{1-x^2}$, then $y^2 = 1 - x^2$, so $x^2 + y^2 = 1$. Since $y = \sqrt{1-x^2} \geq 0$ for $-1 \leq x \leq 1$, it follows that the region in question is the one bounded above by the upper unit semicircle, *i.e.* $y = \sqrt{1-x^2}$, and below by the x-axis, *i.e.* $y = 0$, for $-1 \leq x \leq 1$.

Here's a sketch:



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c. Find the area of the region bounded by $y = x^3 - x$ and $y = 3x$.

Solution. We first need to determine where $y = x^3 - x$ and $y = 3x$ intersect. Setting $x^3 - x = 3x$, which amounts to $x^3 = 4x$, we see that $x = 0$ must be one solution. The solutions with $x \neq 0$ are given by $x^2 = 4$, *i.e.* $x = -2$ and $x = 2$.

We now need to determine when one curve is above the other, which we do by testing points between the points where the two are equal. Note that $-2 < -1 < 0$ and $0 < 1 < 2$. Plugging in $x = -1$ into both equations gives $y = (-1)^3 - (-1) = 0$ and $y = 3(-1) = -3$, so between $x = -2$ and $x = 0$, $y = x^3 - x$ is above $y = 3x$. Plugging in $x = 1$ into both equations gives $y = 1^3 - 1 = 0$ and $y = 3 \cdot 1 = 3$, so between $x = 0$ and $x = 2$, $y = 3x$ is above $y = x^3 - x$.

It follows that the area in question is given by:

$$
\begin{aligned}
&\int_{-2}^{0} \left((x^3 - x) - 3x \right) dx + \int_{0}^{2} \left(3x - (x^3 - x) \right) dx \\
=&\int_{-2}^{0} \left(x^3 - 4x \right) dx + \int_{0}^{2} \left(4x - x^3 \right) dx \\
=&\left(\frac{1}{4}x^4 - \frac{4}{2}x^2 \right)\Bigg|_{-2}^{0} + \left(\frac{4}{2}x^2 - \frac{1}{4}x^4 \right)\Bigg|_{0}^{2} \\
=&\left(\frac{1}{4} \cdot 0^4 - 2 \cdot 0^2 \right) - \left(\frac{1}{4} \cdot (-2)^4 - 2 \cdot 2^2 \right) + \left(2 \cdot 2^2 - \frac{1}{4} \cdot 2^4 \right) - \left(2 \cdot 0^2 - \frac{1}{4} 0^4 \right) \\
=&(0 - 0) - (4 - 8) + (8 - 4) - (0 - 0) = 8 \quad \blacksquare
\end{aligned}
$$

d. Compute $\displaystyle\int_0^1 (x+1)\,dx$ using the Right-hand Rule.

Solution. The general Right-hand Rule formula is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

Plugging in $a = 0$, $b = 1$, and $f(x) = x + 1$ gives:

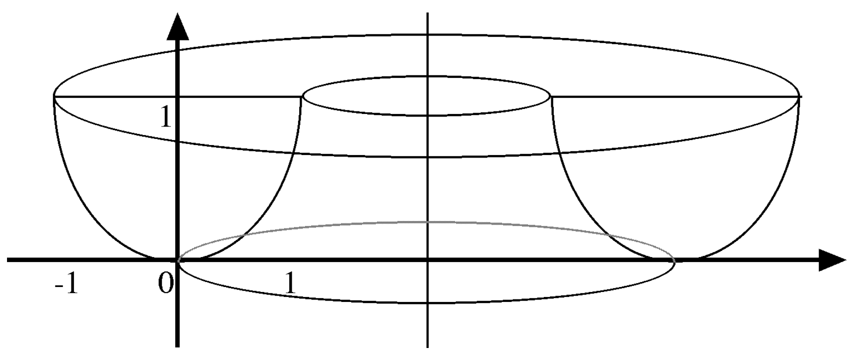
$$\begin{aligned}
\int_0^1 (x+1)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(0 + i\frac{1-0}{n}\right) + 1\right] \cdot \frac{1-0}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[i\frac{1}{n} + 1\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\left(\sum_{i=1}^{n} \frac{i}{n}\right) + \left(\sum_{i=1}^{n} 1\right)\right] = \lim_{n\to\infty} \frac{1}{n} \left[\left(\frac{1}{n}\sum_{i=1}^{n} i\right) + n\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\frac{1}{n} \cdot \frac{n(n+1)}{2} + n\right] = \lim_{n\to\infty} \frac{1}{n} \left[\frac{n+1}{2} + n\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\frac{3}{2}n + \frac{1}{2}\right] = \lim_{n\to\infty} \left[\frac{3}{2} + \frac{1}{2n}\right] = \frac{3}{2} + 0 = \frac{3}{2}
\end{aligned}$$

This computation uses the facts that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{i=1}^{n} 1 = n$ along the way. ■

3. Consider the solid obtained by rotating the region bounded by $y = x^2$ and $y = 1$ about the line $x = 2$.

a. Sketch the solid. *[5]*

Solution.



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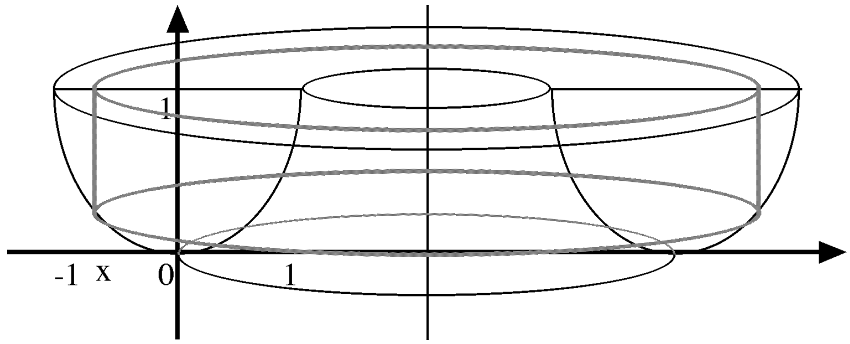
b. Find the volume of the solid. *[20]*

Solution. We will use the method of cylindrical shells to compute the volume of the given solid of revolution. Since we rotated about a vertical line, we need to use x as the variable.

Here is the sketch for part **a** with a cylindrical shell drawn in:



It is easy to see from this diagram that for the cylindrical shell at x, the radius is $r = 2 - x$ and the height is $h = 1 - x^2$.

The volume of the solid is now computed as follows:

$$\begin{aligned}
\int_{-1}^{1} 2\pi r h \, dx &= \int_{-1}^{1} 2\pi(2-x)\left(1-x^2\right)\, dx = 2\pi \int_{-1}^{1} \left(2 - x - 2x^2 + x^3\right)\, dx \\
&= 2\pi \left(2x - \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 \right)\Bigg|_{-1}^{1} \\
&= 2\pi \left(2 \cdot 1 - \frac{1}{2} \cdot 1^2 - \frac{2}{3} \cdot 1^3 + \frac{1}{4} \cdot 1^4 \right) \\
&\qquad - 2\pi \left(2(-1) - \frac{1}{2}(-1)^2 - \frac{2}{3}(-1)^3 + \frac{1}{4}(-1)^4 \right) \\
&= 2\pi \left(2 - \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) - 2\pi \left(-2 - \frac{1}{2} + \frac{2}{3} + \frac{1}{4} \right) \\
&= \cdots = \frac{16\pi}{3} \quad \blacksquare
\end{aligned}$$

4. Find the arc-length of the curve given by the parametric equations $x = \dfrac{t^2}{\sqrt{2}}$ and $y = \dfrac{t^3}{3}$, where $0 \le t \le 1$. *[15]*

Solution. Note that $\dfrac{dx}{dt} = \dfrac{2t}{\sqrt{2}} = \sqrt{2}t$ and $\dfrac{dy}{dt} = \dfrac{3t^2}{3} = t^2$. The arc-length is then:

$$\begin{aligned}
\int_0^1 ds &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{\left(\sqrt{2}t\right)^2 + \left(t^2\right)^2}\, dt \\
&= \int_0^1 \sqrt{2t^2 + t^4}\, dt = \int_0^1 \sqrt{t^2\left(2 + t^2\right)}\, dt = \int_0^1 t\sqrt{2 + t^2}\, dt
\end{aligned}$$

We use the substitution $u = 2 + t^2$, so $du = 2t\,dt$ and $\frac{1}{2}\,du = t\,dt$. Also, $u = 2$ when $t = 0$ and $u = 3$ when $t = 1$.

$$= \int_2^3 \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_2^3 = \frac{1}{3}\cdot 3^{3/2} - \frac{1}{3}\cdot 2^{3/2}$$

$$= \frac{1}{3}\cdot 3\sqrt{3} - \frac{1}{3}\cdot 2\sqrt{2} = \sqrt{3} - \frac{2}{3}\sqrt{2} \quad \blacksquare$$

[Total = 100]