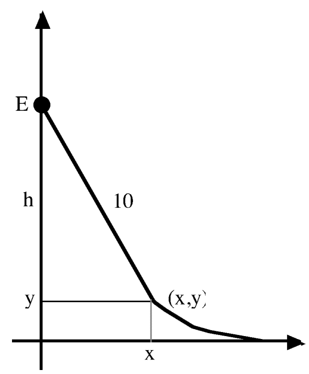
Mathematics 110 – Calculus of one variable

Trent University 2002-2003

SOLUTIONS TO ASSIGNMENT #8

1. The doglets Solovey and Elvis are playing in the Cartesian coordinate system.† Elvis is at the origin and Solovey is 10 m away on the positive x-axis along when Elvis starts running at a constant speed of 10 m/s along the positive y-axis. Solovey gives chase and runs directly towards Elvis at any given instant but always remains 10 m away.‡ Find an equation in x and y for the curve along which Solovey runs. *[8]*

Hint: At any given instant the line segment between the positions of Solovey and Elvis is 10 m long and tangent to the curve along which Solovey runs.



Solution. The key to the solution lies in the geometry of the problem. Consider the right triangle in the slightly augmented diagram above, the one with vertices E, $(0, y)$, and (x, y), where E is Elvis' position and (x, y) is Solvey's position at the same instant. Our task is to determine y as a function of x.

From the information in the problem, the hypotenuse of this triangle is 10 m long and is tangent to Solovey's path. The latter fact implies that the slope of the hypotenuse is $\frac{dy}{dx}$ and that this equals $-\frac{h}{x}$. Since the Pythagorean theorem implies that $h^2+x^2 = 10^2 = 100$, so $h = \sqrt{100 - x^2}$, it follows that $\frac{dy}{dx} = -\frac{1}{x}\sqrt{100 - x^2}$. Hence, by the fundamental theorem of calculus, up to a constant:

$$y = \int -\frac{1}{x}\sqrt{100 - x^2}\, dx$$

† The doglets are real; the yard has been changed to protect the innocent.

‡ Elvis is just a bit faster than Solovey ...

We will integrate this using the trig substitution $x = 10\sin(t)$, so $dx = 10\cos(t)\,dt$:

$$
\begin{aligned}
y &= \int -\frac{1}{x}\sqrt{100-x^2}\,dx = -\int \frac{1}{10\sin(t)}\sqrt{100-(10\sin(t))^2}\,10\cos(t)\,dt \\
&= -\int \frac{\cos(t)}{\sin(t)} 10\sqrt{1-\sin^2(t)}\,dt = -10\int \frac{\cos(t)}{\sin(t)}\sqrt{\cos^2(t)}\,dt \\
&= -10\int \frac{\cos(t)}{\sin(t)}\cos(t)\,dt = -10\int \frac{\cos^2(t)}{\sin(t)}\,dt \\
&= -10\int \frac{1-\sin^2(t)}{\sin(t)}\,dt = -10\int \left(\frac{1}{\sin(t)} - \sin(t)\right)\,dt \\
&= -10\int \left(\csc(t) - \sin(t)\right)\,dt = 10\int \left(\sin(t) - \csc(t)\right)\,dt \\
&= 10\left(\int \sin(t)\,dt - \int \csc(t)\,dt\right) = 10\left(\cos(t) - \left[-\ln\left(\cot(t)+\csc(t)\right)\right]\right) + C \\
&= 10\left(\sqrt{1-\left(\frac{x}{10}\right)^2} + \ln\left(\frac{\cos(t)}{\sin(t)} + \frac{1}{\sin(t)}\right)\right) + C \\
&= 10\sqrt{1-\left(\frac{x}{10}\right)^2} + 10\ln\left(\frac{\sqrt{1-\left(\frac{x}{10}\right)^2}}{\frac{x}{10}} + \frac{1}{\frac{x}{10}}\right) + C \\
&= \sqrt{100-x^2} \;+\; 10\ln\left(\frac{10}{x}\sqrt{1-\left(\frac{x}{10}\right)^2} + \frac{10}{x}\right) + C \\
&= \sqrt{100-x^2} \;+\; 10\ln\left(\frac{1}{x}\sqrt{100-x^2} + \frac{10}{x}\right) + C
\end{aligned}
$$

It remains to determine the constant C. This can be done by using the fact that Solovey started at the point $(10, 0)$, *i.e.* that $y = 0$ when $x = 0$. We can plug these values into the equation above and solve for C:

$$
\begin{aligned}
0 &= \sqrt{100-10^2} \;+\; 10\ln\left(\frac{1}{10}\sqrt{100+10^2} - \frac{10}{10}\right) + C \\
&= 0 + 10\ln\left(\frac{1}{10}\cdot 0 + 1\right) + C \\
&= 10\ln(1) + C = 10\cdot 0 + C = 0 + C = C
\end{aligned}
$$

It follows that the equation of Solovey's path is

$$
y = \sqrt{100-x^2} \;+\; 10\ln\left(\frac{1}{x}\sqrt{100-x^2} + \frac{10}{x}\right).
$$

Whew! ■

2. Four beetles – A, B, C, and D – occupy the corners of a square 10cm along a side. Simultaneously, A begins to crawl directly towards B, B towards C, C towards D, and D towards A. If all four beetles crawl at the same constant rate, their paths will be four congruent spirals that meet in the centre of the square. What distance does each beetle crawl before they all meet? *[2]*

Solution. Here is the solution given by Martin Gardner, from whom I stole this problem**:

> At any given instant the four beetles form the corners of a square which shrinks and rotates as the beetles move closer together. The path of each pursuer will therefore be perpendicular to the path of the pursued. This tells us that as A, for example, approaches B, there is no component in B's motion which carries B towards or away from A. Consequently A will capture B in the same time that it would take if B had remained stationary. The length of each spiral path will be the same as the side of the square: 10 inches [cm in our version].

Note how a bit of vector algebra sneaks in here in the consideration of the components of B's motion relative to A's. ■

Bonus. The 12th Century A.D. Indian mathematician Bhaskara wrote much of his work in verse. Here is one of his problems*:

> The square root of half the number of bees in a swarm
> Has flown out upon a jasmine bush;
> Eight ninths of the swarm has remained behind;
> And a female bee flies about a male who is buzzing inside a lotus flower;
> In the night, allured by the flower's sweet odour, he went inside it
> And now he is trapped!
> Tell me, most enchanting lady, the number of bees.

Restate the problem posed by Bhaskara as an equation and solve it. *[1]*

Solution. Let x be the number of bees in the swarm. Of these x bees, $\frac{8}{9}$ stayed behind, $\sqrt{\frac{x}{2}}$ flew out upon a jasmine bush, and 2 ended up in or near the lotus flower. This gives the equation $x = \frac{8}{9}x + \sqrt{\frac{x}{2}} + 2$. To solve it, we do some algebra:

$$x = \frac{8}{9}x + \sqrt{\frac{x}{2}} + 2 \iff \frac{1}{9}x - 2 = \sqrt{\frac{x}{2}} \iff \frac{1}{81}x^2 - \frac{4}{9}x + 4 = \frac{x}{2}$$
$$\iff \frac{1}{81}x^2 - \frac{17}{18}x + 4 = 0 \iff 2x^2 - 153x + 648 = 0$$

Throwing the quadratic formula at this equation (details left to you!) gives the solutions $\frac{9}{2}$ and 72. Unless Eric the Half-a-Bee is in the swarm, the number of bees must be an integer, so there are 72 bees in the swarm. ■

** *Mathematical Puzzles and Diversions*, Martin Gardner, Penguin Books (1965), p. 110. This book is a collection of some of Gardner's columns in *Scientific American* from the 1950's.

* This translation is given in *The Heritage of Thales*, by W.S. Anglin & J. Lambeck, p. 113.