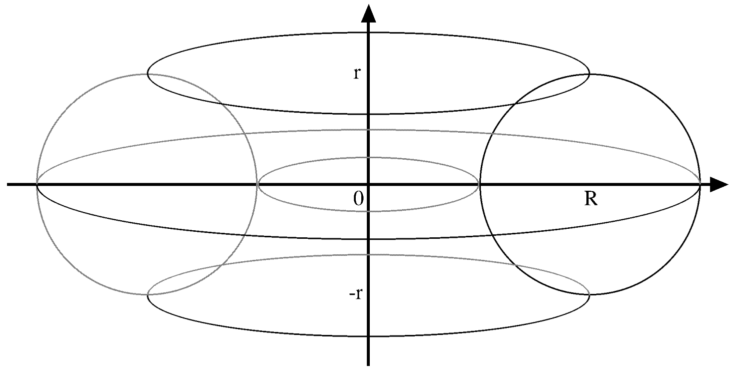
Mathematics 110 – Calculus of one variable

Trent University 2002-2003

SOLUTION TO ASSIGNMENT #7

1. Suppose r and R are constants such that $0 < r < R$. Find the surface area of the torus obtained by rotating the circle $(x-R)^2 + y^2 = r^2$ about the y-axis. *[10]*



Solution. One of the quickest way to solve this is to parametrize the original circle, $(x-R)^2 + y^2 = r^2$. (Note that R and r are constants ...) This circle can be described parametrically as follows: $x = R + r\cos(t)$ and $y = r\sin(t)$, where $0 \le t \le 2\pi$. To check that this works, try plugging these expressions for x and y into the original equation of the circle.

The area of the surface obtained by rotating a curve C about the y-axis is given by $\int_C 2\pi x\, ds$, where ds is an infinitesimal piece of arc-length of the curve. In the case of a curve given by parametric equations $x = x(t)$ and $y = y(t)$ for $\alpha \le t \le \beta$, this integral boils down to (see §10.3 in the text):

$$\int_\alpha^\beta 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

In our case, $x = R + r\cos(t)$, so $\dfrac{dx}{dt} = 0 - r\sin(t)$, and $y = r\sin(t)$, so $\dfrac{dy}{dt} = r\cos(t)$.

It follows that the surface area we want can be computed as follows:

$$\begin{aligned}
\int_0^{2\pi} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= \int_0^{2\pi} 2\pi\left(R + r\cos(t)\right)\sqrt{\left(-r\sin(t)\right)^2 + \left(r\cos(t)\right)^2}\, dt \\
&= \int_0^{2\pi} 2\pi\left(R + r\cos(t)\right)\sqrt{r^2\left(\sin^2(t) + \cos^2(t)\right)}\, dt \\
&= \int_0^{2\pi} 2\pi\left(R + r\cos(t)\right)\sqrt{r^2}\, dt \\
&= \int_0^{2\pi} 2\pi r\left(R + r\cos(t)\right)\, dt = 2\pi r\left(Rt - r\sin(t)\right)\Big|_0^{2\pi} \\
&= 2\pi r\left(R\cdot 2\pi - r\sin(2\pi)\right) - 2\pi r\left(R\cdot 0 - r\sin(0)\right) \\
&= 2\pi r\left(2\pi R - r\cdot 0\right) - 2\pi r\left(0 - r\cdot 0\right) \\
&= 2\pi r\left(2\pi R\right) = 4\pi^2 Rr \qquad \blacksquare
\end{aligned}$$