Mathematics 110 – Calculus of one variable

Trent University 2001-2002

SOLUTIONS TO ASSIGNMENT #9

Sets, gaps, and size

We'll look at two examples of infinite sets of real numbers which have a small length. First, consider the following process:

0. Start with the closed unit interval $[0,1]$.
1. Remove the open middle third, *i.e.* $\left(\frac{1}{3},\frac{2}{3}\right)$, leaving $\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]$.
2. Remove the open middle third from each of the intervals remaining, *i.e.* $\left(\frac{1}{9},\frac{2}{9}\right)$ and $\left(\frac{7}{9},\frac{8}{9}\right)$, leaving $\left[0,\frac{1}{9}\right]\cup\left[\frac{2}{9},\frac{1}{3}\right]\cup\left[\frac{2}{3},\frac{7}{9}\right]\cup\left[\frac{8}{9},1\right]$.

$\vdots$

n. Remove the open middle third from each of the intervals remaining from the immediately preceding step.

$\vdots$

The Cantor set is the subset C of the closed unit interval $[0,1]$ which consists of all the points remaining after infinitely many steps of this process have been carried out.

1. Show that the total length of the intervals removed from the closed interval $[0,1]$ to make C is 1. What is the total length of C, not counting the gaps? Why? *[4]*

Solution. At each step the length remaining is two-thirds of the length at the previous step, and number of sub-intervals remaining is double the number at the previous step while their individual length is one-third of those at the previous step. We thus have the following pattern:

step	*length remaining*	*length removed*
0	1	0
1	$\frac{2}{3}$	$\frac{1}{3}$
2	$\frac{4}{9}$	$\frac{1}{3}+\frac{2}{9}$
3	$\frac{8}{27}$	$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}$
4	$\frac{16}{81}$	$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{2^n}{3^n}$	$\frac{1}{3}+\frac{2}{9}+\cdots+\frac{2^{n-1}}{3^n}$
$\vdots$	$\vdots$	$\vdots$

It follows that the length remaining after the process, *i.e.* the length of C, is

$$\lim_{n\to\infty}\frac{2^n}{3^n}=\lim_{n\to\infty}\left(\frac{2}{3}\right)^n=0$$

while the total length of the intervals removed during the process is

$$\sum_{n=1}^{\infty}\frac{2^{n-1}}{3^n}=\sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}=\frac{\frac{1}{3}}{1-\frac{2}{3}}=\frac{\frac{1}{3}}{\frac{1}{3}}=1$$

since what we have here is a geometric series with first term $a = \frac{1}{3}$ and common ratio $r = \frac{2}{3}$. ■

Second, let $Q = [0,1] \cap \mathbb{Q}$ be the set of rational numbers — *i.e.* those that can be written as ratios of integers — in the closed unit interval $[0,1]$. One interesting fact about Q is that it can be *enumerated*: there is a sequence $\{q_n\}$ listing all the elements of Q.

2. Show that the total length, not counting any gaps, of the set $Q = [0,1] \cap \mathbb{Q}$ is 0. *[3]*

Solution. We will show that the length of Q must be less than ε for every $\varepsilon > 0$.

Suppose an $\varepsilon > 0$ is given. First, note that the geometric series with $a = \frac{\varepsilon}{4}$ and $r = \frac{1}{2}$,

$$\sum_{n=0}^{\infty} \frac{\varepsilon}{4}\left(\frac{1}{2}\right)^n = \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{16} + \cdots ,$$

has sum

$$\frac{a}{1-r} = \frac{\frac{\varepsilon}{4}}{1-\frac{1}{2}} = \frac{\frac{\varepsilon}{4}}{\frac{1}{2}} = \frac{\varepsilon}{2} .$$

It follows that the sequence of intervals

$$\begin{gathered} \left(q_1 - \tfrac{\varepsilon}{8}, q_1 + \tfrac{\varepsilon}{8}\right) \\ \left(q_2 - \tfrac{\varepsilon}{16}, q_2 + \tfrac{\varepsilon}{16}\right) \\ \left(q_3 - \tfrac{\varepsilon}{32}, q_3 + \tfrac{\varepsilon}{32}\right) \\ \vdots \\ \left(q_n - \tfrac{\varepsilon}{2^{n+2}}, q_n + \tfrac{\varepsilon}{2^{n+2}}\right) \\ \vdots \end{gathered}$$

has collective length (which would be reduced by any overlaps) at most

$$\frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{16} + \cdots = \frac{\varepsilon}{2} < \varepsilon .$$

Because each q_n is an element of the nth interval, the total length of $Q = \{ q_n \mid n \geq 1 \}$ is less than or equal to the collective length of the intervals, and hence is less than ε.

Since a length must be greater than or equal to 0, and there is only one number greater than or equal to 0 and less than ε for every $\varepsilon > 0$, it follows that the only length Q could be is 0. ■

Alternate Solution. If S is a subset of $[0,1]$, let $[0,1] \setminus S = \{ x \in [0,1] \mid x \notin S \}$ denote the set of all numbers in $[0,1]$ which are *not* in S. We will compute the length of Q by computing the length of $[0,1] \setminus Q$ and then subtracting that from the length of $[0,1]$.

Note that for each n, $[0,1] \setminus \{ q_1, q_2, \ldots, q_n \}$ has length 1. (Why? A complete solution would require an explanation ...) It follows that

$$\text{length of } [0,1] \setminus Q = \lim_{n\to\infty} \text{length of } [0,1] \setminus \{ q_1, q_2, \ldots, q_n \} = \lim_{n\to\infty} 1 = 1 .$$

It follows in turn that the length of Q is 1 minus the length of $[0,1] \setminus Q$, *i.e.* $1 - 1 = 0$. ■

3. Show that if r, $s \in Q$ and $r < s$, then there is a $t \in Q$ such that $r < t < s$. *[1]*

Solution. If r and s are in Q, they are both ratios of integers, say $r = \frac{a}{b}$ and $s = \frac{c}{d}$, for some integers a, b, c, and d. It follows that $t = \frac{r+s}{2}$ does the job since $\frac{r+s}{2} = \frac{ad+bc}{2bd}$ is obviously a ratio of integers and, using the fact that $r < s$,

$$r = \frac{r+r}{2} < \frac{r+s}{2} < \frac{s+s}{2} = s\,.$$

Note that since $0 \le r < t < s \le 1$ and t is a ratio of integers, $t \in Q$. ■

4. Discuss the gaps in Q in light of **2** and **3** ... *[2]*

Solution. It follows from **3** that the gaps in Q are very small: since they cannot include any interval, they must consist of single points, namely the (locations of the) irrational real numbers in $[0,1]$. Each gap thus has length 0. On the other hand, **2** tells us the gaps have a collective length of 1. So a bunch of things of length 0 add up to a length of 1 ... This sort of paradox suggests that "length" is not a simple concept, and does not really behave all that untuitively when applied to complicated sets ...

Unfortunately, to define integration for as many functions as possible, it is necessary to deal with complicated sets. For example, knowing that Q has a length ("measure" to professionals) of 0 allows one to integrate the function

$$f(x) = \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \end{cases},$$

which Riemann integration will not handle:

$$\begin{aligned} \int_0^1 f(x)\,dx &= \int_Q f(x)\,dx + \int_{[0,1]\setminus Q} f(x)\,dx \\ &= \int_Q 1\,dx + \int_{[0,1]\setminus Q} 0\,dx \\ &= 1 \cdot \text{length of } Q + 0 \cdot \text{length of } [0,1] \setminus Q \\ &= 1 \cdot 0 + 0 \cdot 1 = 0 \end{aligned}$$

If you would like to find out more about this sort of thing, look up measure theory in general, and Lebesque measure and integration in particular. ■

Bonus. Show that C has more elements than Q. *[1]*

Solution. C has just as many elements as $[0,1]$, which is too big to be enumerated, unlike Q. Why? Ask if you want to know! ■