Integration by Parts

The Product Rule Rearranged and Reversed

The Product Rule of differential calculus tells us how to compute the derivative of a product of two functions $f(x)g(x)$:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

If we rearrange this equation as $f(x)g'(x) = \frac{d}{dx}\left(f(x)g(x)\right) - f'(x)g(x)$ and integrate on both sides, we get the formula for integration by parts:

$$\begin{aligned}\int f(x)g'(x)\,dx &= \int \frac{d}{dx}\left(f(x)g(x)\right)\,dx - \int f'(x)g(x)\,dx \\ &= f(x)g(x) - \int f'(x)g(x)\,dx\end{aligned}$$

If we write $u = f(x)$ and $v = g(x)$, the original Product Rule looks like $(uv)' = u'v + uv'$ and the integral formula becomes:

Integration by Parts: $$\int uv'\,dx = uv - \int u'v\,dx$$

This is the form most often seen in single variable calculus textbooks. The definite integral form of this is:

Integration by Parts: $$\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b u'v\,dx$$

The usual motive behind the use of integration by parts , as with substitution, is to simplify the integrand you have to deal with. That is, one decomposes the original integrand into a product, one component of which is the u and the other of which is the v', and one does so in such a way as to make the remaining integral easier to handle. This is best seen by working through some examples.

Example 1. Suppose we wish to integrate $\int xe^x\,dx$. We need to decide which part of the integrand will be u and which will be v'. There are two obvious possibilities, make $u = e^x$ and $v' = x$, or the the other way around.

If we go with $u = e^x$ and $v' = x$, we need to compute u' and v to apply the integration by parts formula: $u' = \dfrac{d}{dx}e^x = e^x$ and $v = \displaystyle\int v'\,dx = \int x\,dx = \frac{x^2}{2}$. (We don't worry about the generic C in computing v; with definite integrals it would cancel out anyway and with indefinite integrals we won't be putting in a generic constant until the last integral sign goes away.) Plugging these into the formula gives us:

$$\int xe^x\,dx = e^x \cdot \frac{x^2}{2} - \int e^x \cdot \frac{x^2}{2}\,dx$$

The integral we still have to deal with is now more complicated than the one we started with, which is a sign we chose poorly in deciding which part of the original integrand was to be u and which was to be v'.

Starting over, suppose we decide to have $u = x$ and $v' = e^x$ in trying to apply integration by parts to $\displaystyle\int xe^x\,dx$. Then $u' = \dfrac{d}{dx}x = 1$ and $v = \displaystyle\int e^x\,dx = e^x$. Plugging these into the integration by parts formula gives:

$$\int xe^x\,dx = xe^x - \int 1e^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

Since we are trying to compute an indefinite integral, once the last integral sign has finally disappeared it's time to put in the generic constant of integration.

The initial false step in the example above illustrates one of the main pitfalls in trying to use integration by parts: choosing poorly which part of the integrand is to be u and which is to be v' can be counterproductive. While it may not work all the time the following rule of thumb is very handy a lot of the time in making these selection:

> *If you have an integrand that is a product of two different types of function, put whichever appears first on the list below into u when using integration by parts, with the rest of the integrand going into v':*
>
> *logarithmic (including inverse hyperbolic)*
> *inverse trigonometric*
> *polynomials (and a lot of functions not otherwise on this list)*
> *trigonometric*
> *exponential (including hyperbolic)*

If we had applied this rule of thumb in the example above to start with, we would have tried the useful partition of the integrand xe^x as $u = x$ and $v' = e^x$ first.

A more concise rule of thumb that is widely applicable, but which may require some compromises or experimentation in practice, is the following:

> *If at all possible, try to select u and v' so that u' is simpler than u and v no worse than v'.*

The example above illustrates this rule of thumb as well: selecting $u = x$ makes $u' = 1$ simpler, while $v = e^x$ is no worse than $v' = e^x$; however, selecting $u = e^x$ does not make $u' = e^x$ any simpler, while $v = \dfrac{x^2}{2}$ is more complex than $v = x$.

Example 2. Let's try to compute $\int_1^e \ln(x)\,dx$. Being just $\ln(x)$, this integrand has the problem that it's not immediately obvious how to decompose it as a product. The only easy way to do so is the trivial $\ln(x) = 1 \cdot \ln(x)$; following the first rule of thumb, we decide to make $u = \ln(x)$ and $v' = x$. Then $u' = \dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$ and $v = \displaystyle\int 1\,dx = x$. (Note that this means that we've compromised on the second rule of thumb.) Plugging these into the inegration by parts formula for definite integrals gives us:

$$\begin{aligned}\int_1^e \ln(x)\,dx &= \ln(x)\cdot x\big|_1^e - \int_1^e \frac{1}{x}\cdot x\,dx \\ &= x\ln(x)\big|_1^e - \int_1^e 1\,dx = x\ln(x)\big|_1^e - x\big|_1^e \\ &= (e\ln(e) - 1\ln(1)) - (e-1) \\ &= (e\cdot 1 - 1\cdot 0) - e + 1 = e - 0 - e + 1 = 1\end{aligned}$$

A similar trick can be used to compute $\int \arctan(x)\,dx$ if you want to try something along these lines for practice.

Annoyingly, there are situations where we may have to use integration by parts more than once in the same problem.

Example 3. Let's try to integrate $\int x^2 cos(x)\,dx$. Either rule of thumb would suggest trying $u = x^2$ and $v' = \cos(x)$, which gives $u' = \frac{d}{dx}x^2 = 2x$ and $v = \int \cos(x)\,dx = \sin(x)$. Plugging these into the inegration by parts formula gives:

$$\int x^2 cos(x)\,dx = x^2\sin(x) - \int 2x\sin(x)\,dx$$

The remaining integral, $\int 2x\sin(x)\,dx$ still needs to be sorted out. Following either rule of thumb suggests trying $s = 2x$ and $t' = \sin(x)$ [we've already used u and v in this example and to do so again in a different way risks confusion], so $s' = \frac{d}{dx}(2x) = 2$ and $t = \int \sin(x)\,dx = -\cos(x) = (-1)\cos(x)$. Plugging these into the integration by parts formula gives:

$$\begin{aligned}\int 2x\sin(x)\,dx &= 2x\cdot(-1)\cos(x) - \int 2(-1)\cos(x)\,dx \\ &= -2x\cos(x) + \int 2\cos(x)\,dx \\ &= -2x\cos(x) + 2\sin(x)\end{aligned}$$

Plugging this back into the original equation lets us finish the original job:

$$\begin{aligned}\int x^2 cos(x)\,dx &= x^2\sin(x) - \int 2x\sin(x)\,dx \\ &= x^2\sin(x) - [-2x\cos(x) + 2\sin(x)] + C \\ &= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C\end{aligned}$$

Note that because this is and indefinite integral, once the last integral sign is gone it is time for the generic constant of integration, C, to appear.

Writing the solution with lots of explanations, especially breaking out an easy subsidiary calculation, is a little tedious and inefficient in practice, so most people would probably write this one up as something like:

$$\begin{aligned}
\int x^2 cos(x)\,dx &= x^2\sin(x) - \int 2x\sin(x)\,dx \qquad \begin{array}{l}\text{Using parts } u = x^2 \text{ and } v' = \cos(x),\\ \text{so } u' = 2x \text{ and } v = \sin(x).\end{array}\\
&= x^2\sin(x) - \left[2x\cdot(-1)\cos(x) - \int 2(-1)\cos(x)\,dx\right]\\
&\qquad \text{Using parts } s = 2x \text{ and } t' = \sin(x), \text{ so } s' = 2 \text{ and } t = -\cos(x).\\
&= x^2\sin(x) - [2x\cdot(-1)\cos(x) - 2(-1)\sin(x)] + C\\
&= x^2\sin(x) - [-2x\cos(x) + 2\sin(x)] + C\\
&= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C
\end{aligned}$$

An interesting variation on using parts repeatedly in a problem can occur if both parts are functions that don't really get simpler whether you integrate or differentiate them.

Example 4. Let's try to compute $\displaystyle\int e^x\sin(x)\,dx$ using integration by parts. The second rule of thumb isn't much use here because whether you integrate or differentiate either of e^x or $\sin(x)$, you get a function of the same type and complexity. However, the first rule of thumb suggests that we try $u = \sin(x)$ and $v' = e^x$ because trig functions come before exponential functions on the list. Let's go with this:

$$\begin{aligned}
\int e^x\sin(x)\,dx &= e^x\sin(x) - \int e^x\cos(x)\,dx \qquad \begin{array}{l}u = \sin(x) \text{ and } v' = e^x,\\ u' = \cos(x) \text{ and } v = e^x.\end{array}\\
&= e^x\sin(x) - \left[e^x\cos(x) - \int e^x\left(-\sin(x)\right)\,dx\right] \qquad \begin{array}{l}s = \cos(x) \text{ and } t' = e^x,\\ s' = -\sin(x) \text{ and } t = e^x.\end{array}\\
&= e^x\sin(x) - \left[e^x\cos(x) + \int e^x\sin(x)\,dx\right]\\
&= e^x\sin(x) - e^x\cos(x) - \int e^x\sin(x)\,dx
\end{aligned}$$

This boils down to $\displaystyle\int e^x\sin(x)\,dx = e^x\sin(x) - e^x\cos(x) - \int e^x\sin(x)\,dx$, which we can solve for $\int e^x\sin(x)\,dx$ by treating the whole integral as an unknown in an equation:

$$\begin{aligned}
&\int e^x\sin(x)\,dx = e^x\sin(x) - e^x\cos(x) - \int e^x\sin(x)\,dx\\
\implies\quad & 2\int e^x\sin(x)\,dx = e^x\sin(x) - e^x\cos(x)\\
\implies\quad & \int e^x\sin(x)\,dx = \frac{1}{2}e^x\sin(x) - \frac{1}{2}e^x\cos(x) + C
\end{aligned}$$

We delayed adding the generic constant of integration until we isolated the integral we started with.