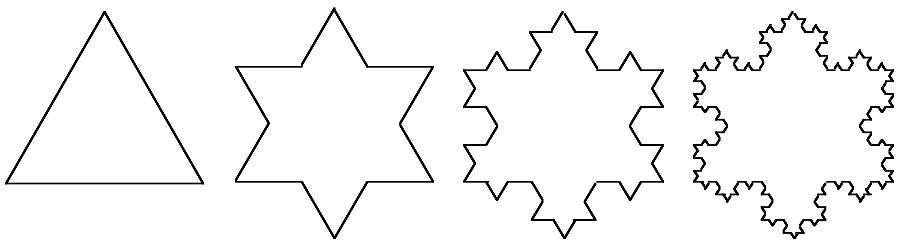
Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2024

Solutions to Assignment #7
A Snowflake Fence

A shape that looks something like a snowflake is defined by the following process:

> *Start with an equilateral triangle whose sides have length 1. If one modifies each of the line segments composing the triangle by cutting out the middle third of the segment, and then inserting an outward-pointing "tooth," both of whose sides are as long as the removed third, one gets a six-pointed star. Suppose one repeats this process for each of the line segments making up the star, then to each of the line segments making up the resulting figure, and so on. The first few steps are illustrated below:*



The snowflake shape is the limit of this process; *i.e.* what you have after infinitely many steps.

1. What is the length of the perimeter of the snowflake shape? *[4]*

Hint: How does the length of the perimeter change at each step of the process?

SOLUTION. At each step of the process every line segment is replaced by four line segments that are each one third of the length of the line segment they are replacing. This means that ate each step, the perimeter of the shape increases in length by a factor of $\frac{4}{3}$. Since the equilateral triangle at the beginning, *i.e.* at step 0, has sides of length 1, it has perimeter length 3. It follows that at step $n \geq 0$ the length of the perimeter is $3\left(\frac{4}{3}\right)^n$, and so the snowflake shape has a perimeter of length $\lim_{n\to\infty} 3\left(\frac{4}{3}\right)^n = \infty$; it grows without bound because $\frac{4}{3} > 1$.

For the sceptical, here is a more rigorous argument. For any $M > 0$, we can get $3\left(\frac{4}{3}\right)^n \geq M$ for large enough n because

$$3\left(\frac{4}{3}\right)^n \geq M \iff \left(\frac{4}{3}\right)^n \geq \frac{M}{3} \iff n\ln\left(\frac{4}{3}\right) = \ln\left(\left(\frac{4}{3}\right)^n\right) \geq \ln\left(\frac{M}{3}\right) \iff n \geq \frac{\ln\left(\frac{M}{3}\right)}{\ln\left(\frac{4}{3}\right)}.$$

This means that $\lim_{n\to\infty} 3\left(\frac{4}{3}\right)^n$ must be ∞, since $3\left(\frac{4}{3}\right)^n$ eventually exceeds every possible finite upper bound.

One could also use SageMath to compute the limit. □

2. What is the area of the snowflake shape? *[6]*

Hint: What is the area of the original triangle? How much area is added at each step? Can you add up all these areas?

NOTE. You may the find the following summation formula, already seen on Assignment #1, useful in doing **2**. Recall that as long as $|r| < 1$:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$$

The requirement that $|r| < 1$ is necessary.

SOLUTION. I'll leave it to you to check that an equilateral triangle with sides of length s has an area of $\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \cdot s \cdot \frac{\sqrt{3}}{2} s = \frac{\sqrt{3}}{4} s^2$.

At step 0 we have an equilateral triangle with sides of length 1 and hence area $\frac{\sqrt{3}}{4} = \approx 0.4330$. Note that at step 0, the perimeter is made up of 3 line segments, each of length 1.

In step 1 we add the area of three smaller triangles, each with sides of length $\frac{1}{3}$ and hence area $\frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3}\right)^2$. Since there are three of them, this means the total area added at step 1 is $3 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3}\right)^2 = \left(\frac{4}{9}\right)^0 \cdot \frac{1}{3} \cdot \frac{\sqrt{3}}{4} = \left(\frac{4}{9}\right)^0 \cdot \frac{\sqrt{3}}{12}$. Thus the area of the six-pointed star at step 1 is $\frac{\sqrt{3}}{4} + \left(\frac{4}{9}\right)^0 \cdot \frac{\sqrt{3}}{12}$. (Why are we bothering with $\left(\frac{4}{9}\right)^0 = 1$ in this expression? Because that's what fits the pattern that develops below.) Note that at step 1, the perimeter is made up of $3 \cdot 4 = 12$ line segments, each of length $\frac{1}{3}$.

In step 2 we add a small triangle for each of the line segments we had in stage 1. Each of these has sides of length $\frac{1}{9} = \left(\frac{1}{3}\right)^2$. Each thus has area $\frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)^2$, for a total added area of $12 \cdot \left(\frac{1}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \left(\frac{4}{9}\right)^1 \cdot \frac{\sqrt{3}}{12}$. Thus the area of the shape at step 2 is $\frac{\sqrt{3}}{4} + \left(\frac{4}{9}\right)^0 \cdot \frac{\sqrt{3}}{12} + \left(\frac{4}{9}\right)^1 \cdot \frac{\sqrt{3}}{12}$. Note that at step 2, the perimeter is made up of $3 \cdot 4^2 = 48$ line segments, each of length $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$.

In step 3 we add a small triangle for each of the line segments we had in stage 2. Each of these has sides of length $\frac{1}{27} = \left(\frac{1}{3}\right)^3$. Each thus has area $\left(\frac{1}{9}\right)^3 \cdot \frac{\sqrt{3}}{4}$, for a total added area of $48 \cdot \left(\frac{1}{9}\right)^3 \cdot \frac{\sqrt{3}}{4} = \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{12}$. Thus the area of the shape at step 3 is $\frac{\sqrt{3}}{4} + \left(\frac{4}{9}\right)^0 \cdot \frac{\sqrt{3}}{12} + \left(\frac{4}{9}\right)^1 \cdot \frac{\sqrt{3}}{12} + \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{12}$.

In general, at step $n-1$ of the process we have $3 \cdot 4^{n-1}$ line segments, for each of which we add a small triangle with side length $\left(\frac{1}{3}\right)^n$, so each little triangle added at step n has area $\left(\left(\frac{1}{3}\right)^n\right)^2 \cdot \frac{\sqrt{3}}{4} = \left(\frac{1}{3}\right)^{2n} \cdot \frac{\sqrt{3}}{4} = \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4}$. Hence the area *added* to the region at step $n \geq 1$ is:

$$\begin{aligned}
&(\#\text{ triangles added at step } n) \times (\text{area of each triangle}) \\
&= (\#\text{ line segments at step } n-1) \times (\text{length of side})^2 \cdot \frac{\sqrt{3}}{4} \\
&= 3 \cdot 4^{n-1} \times \left(\left(\frac{1}{3}\right)^n\right)^2 \cdot \frac{\sqrt{3}}{4} = 3 \cdot \frac{4^n}{4} \times \frac{1}{9^4} \cdot \frac{\sqrt{3}}{4} = \left(\frac{4}{9}\right)^n \cdot \frac{3\sqrt{3}}{4^2} = \left(\frac{4}{9}\right)^{n-1} \cdot \frac{\sqrt{3}}{12}
\end{aligned}$$

It follows that the total area snowflake shape is

$$\begin{aligned}
\text{Area} &= \frac{\sqrt{3}}{4} + \left(\frac{4}{9}\right)^0 \cdot \frac{\sqrt{3}}{12} + \left(\frac{4}{9}\right)^1 \cdot \frac{\sqrt{3}}{12} + \cdots + \left(\frac{4}{9}\right)^n \cdot \frac{\sqrt{3}}{12} + \cdots \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \cdot \left[1 + \left(\frac{4}{9}\right)^1 + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^n + \cdots\right] \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \cdot \frac{1}{1-\frac{4}{9}} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \cdot \frac{9}{5} = \frac{5\sqrt{3}}{20} + \frac{3\sqrt{3}}{20} = \frac{8\sqrt{3}}{20} = \frac{2\sqrt{3}}{5} \approx 0.6928,
\end{aligned}$$

using the formula for the sum of a geometric series, which is applicable since in this case we have a common ratio of $r = \frac{4}{9}$, which has an absolute value less than 1.

One could also use SageMath to compute the sum. ∎