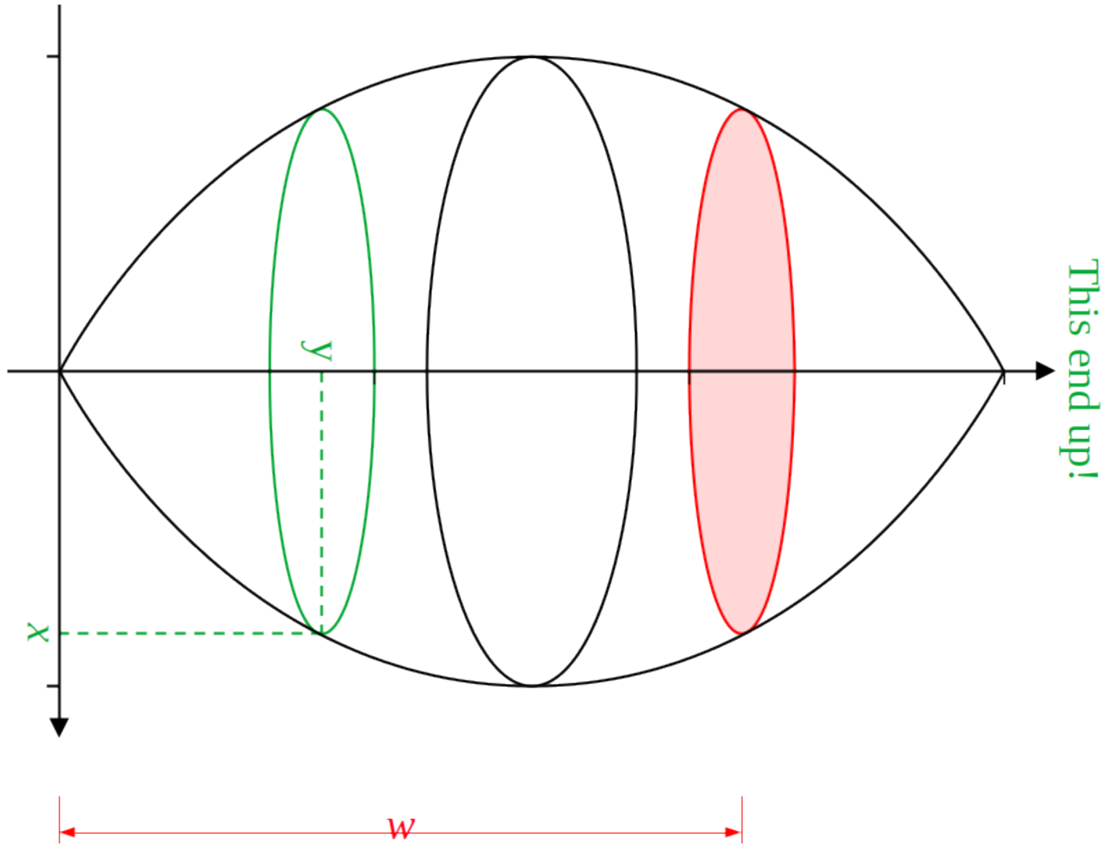
Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2024

Solutions to Assignment #2
Volumes and Rates

A tank has the shape of the solid of revolution obtained by revolving the curve $x = \sin\left(\frac{\pi y}{3}\right)$, for ~~$0 \le x \le 3$~~ $0 \le y \le 3$, about the y-axis, with both axes being measured in metres.



The tank is completely filled with water which is then drained from the tank at a contant rate of 100 litres per minute. Suppose that a given instant the water in the tank is w metres deep.

1. What is the volume (in litres) of the water in the tank at the given instant? Work it out both by hand and by using SageMath. *[6]*

Note. If you don't remember or didn't see how to compute the volume of a solid of revolution, please check out §9.3 in the textbook, or the lectures on this topic from past iterations of MATH 1110H and 1120H on the archive page at: `http://euclid.trentu.ca/math/sb/calculus/`

Solution. We will use the disk/washer method for computing the volume of a solid of revolution. The volume of the water in the tank when the water is w metres deep is obtained by revolving the region given by $0 \le x \le \sin\left(\frac{\pi y}{3}\right)$, for $0 \le x \le w$, about the y-axis. A horizontal cross-section of this shape at y is a disk with radius $r = x - 0 = \sin\left(\frac{\pi y}{3}\right)$, as in the annotated diagram above, and hence area $A(y) = \pi r^2 = \pi \sin^2\left(\frac{\pi y}{3}\right)$. It follows that the volume is volume of the water in the tank when the water is w metres deep is given by $V(w) = \int_0^w A(y)\, dy = \int_0^w \pi \sin^2\left(\frac{\pi y}{3}\right) dy$. We proceed to evaluate this integral.

By hand. We will first simplify the integral by using the substitution $t = \frac{\pi y}{3}$, so $dt = \frac{\pi}{3}\,dy$ and $\pi\,dy = 3\,dt$, changing the limits as we go along: $\begin{matrix} y & 0 & w \\ t & 0 & \pi w/3 \end{matrix}$. The simplified integral will then be computed with the help of the following trigonometric integral reduction formula, given in class: $\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$. Here we go:

$$\begin{aligned}
V(w) &= \int_0^w \pi \sin^2\left(\frac{\pi y}{3}\right)\,dy = \int_0^{\pi w/3} \sin^2(t)\cdot 3\,dt = 3\int_0^{\pi w/3} \sin^2(t)\,dt \\
&= 3\left[-\frac{1}{2}\sin^{2-1}(t)\cos(t)\Big|_0^{\pi w/3} + \frac{2-1}{2}\int_0^{\pi w/3} \sin^{2-2}(t)\,dt\right] \\
&= -\frac{3}{2}\sin(t)\cos(t)\Big|_0^{\pi w/3} + \frac{3}{2}\int_0^{\pi w/3} \sin^0(t)\,dt \\
&= \left(-\frac{3}{2}\sin\left(\frac{\pi w}{3}\right)\cos\left(\frac{\pi w}{3}\right)\right) - \left(-\frac{3}{2}\sin(0)\cos(0)\right) + \frac{3}{2}\int_0^{\pi w/3} 1\,dt \\
&= -\frac{3}{2}\sin\left(\frac{\pi w}{3}\right)\cos\left(\frac{\pi w}{3}\right) + \frac{3}{2}\cdot 0\cdot 1 + \frac{3}{2}t\Big|_0^{\pi w/3} \\
&= -\frac{3}{2}\sin\left(\frac{\pi w}{3}\right)\cos\left(\frac{\pi w}{3}\right) + 0 + \frac{3}{2}\cdot\frac{\pi w}{3} - \frac{3}{2}\cdot 0 \\
&= -\frac{3}{2}\sin\left(\frac{\pi w}{3}\right)\cos\left(\frac{\pi w}{3}\right) + \frac{\pi w}{2} - 0 = \frac{\pi w}{2} - \frac{3}{2}\sin\left(\frac{\pi w}{3}\right)\cos\left(\frac{\pi w}{3}\right)
\end{aligned}$$

This could also have been done by using the identity $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$ and another small subsitution. Doing this would give an answer in a form like that produced by SageMath below.

Using SageMath. The simplest and most naive approach is to hand SageMath the indefinite integral:

```
[1]: var('y')
     integral( pi*(sin(pi*y/3))^2, y )

[1]: 1/2*pi*y - 3/4*sin(2/3*pi*y)
```

and then plug in w and 0 for y in the antiderivative and take the difference. Of course, SageMath can also evaluate definite integrals:

```
[2]: var('w')
     assume(w>0)
     integral( pi*(sin(pi*y/3))^2, y, 0, w )

[2]: 1/2*pi*w - 3/4*sin(2/3*pi*w)
```

Omitting declaring w here results in error messages as SageMath's symbolic integration module does not know what to do with it. Omitting the `assume(w>0)` results in a long sequence of error messages, the last of which advises trying `assume(w>0)` ...

Note that the answer SageMath provides looks a little different from what we obtained by hand above. They are actually equal, via the double angle formula $\sin(2x) = 2\sin(x)\cos(x)$ with $x = \frac{\pi w}{3}$.

It's not over - we have units to contend with! Since we've been measuring lengths in metres, our volume computation is in terms of metres cubed. However we were asked to deliver an answer in litres. There are 1000 litres in a cubic metre, so our final answer should be

$$\begin{aligned} V(w) &= 1000\left[\frac{\pi w}{2} - \frac{3}{2}\sin\left(\frac{\pi w}{3}\right)\cos\left(\frac{\pi w}{3}\right)\right] \\ &= 500\pi w - 1500\sin\left(\frac{\pi w}{3}\right)\cos\left(\frac{\pi w}{3}\right) \text{ litres} \end{aligned}$$

if we go with the hand-done solution, or

$$\begin{aligned} V(w) &= 1000\left[\frac{\pi w}{2} - \frac{3}{4}\sin\left(\frac{2\pi w}{3}\right)\right] \\ &= 500\pi w - 750\sin\left(\frac{2\pi w}{3}\right) \text{ litres} \end{aligned}$$

if we go with the one done by SageMath. □

2. How is the depth of the water in the tank changing at the instant that the depth is 2 metres? Work it out *without* implicitly or explicitly using your final answer to question **1**. You may use SageMath, or do it by hand, or mix these up. *[4]*

Solution. We are given that the tank is being drained at a constant rate of 100 litres or $\frac{100}{1000} = 0.1$ cubic metres per minute, *i.e.* $\frac{dV}{dt} = -0.1\ m^3/min$, and we are asked to find the rate at which w is changing, *i.e.* $\frac{dw}{dt}$. We have V as a function of w, so we have $\frac{dV}{dt} = \frac{dV}{dw} \cdot \frac{dw}{dt}$ by the Chain Rule. This means that we need $\frac{dV}{dw}$ when $w = 2$.

It would be easy enough to get $\frac{dV}{dt}$ by differentiating the volume formulas obtained in solving question **1** and then plugging in $w = 2$, but we are explicitly not allowed to do so. This means we have to do something even easier! The Fundamental Theorem of Calculus tells us that integration and differentiation are opposite operations. In particular, the Theorem's second form tells that if $V(w) = \int_0^w \pi \sin^2\left(\frac{\pi y}{3}\right) dy$, then $\frac{dV}{dw} = \pi \sin^2\left(\frac{\pi w}{3}\right)$, and so

$$\left.\frac{dV}{dw}\right|_{w=2} = \pi \sin^2\left(\frac{\pi \cdot 2}{3}\right) = \pi\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3\pi}{4}.$$

Thus, since $\frac{dV}{dt} = \frac{dV}{dw} \cdot \frac{dw}{dt}$ as noted above:

$$\frac{dw}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dw}} = \frac{-0.1}{\frac{3\pi}{4}} = \frac{-0.4}{3\pi} \approx -0.0424\ m/min$$

Thus the depth of the water in the tank is declining at a rate of about 0.0424 m/min (*i.e.* 4.24 cm/min) at the instant that the water in the tank is 2 metres deep. ■